The Lorenz Transform

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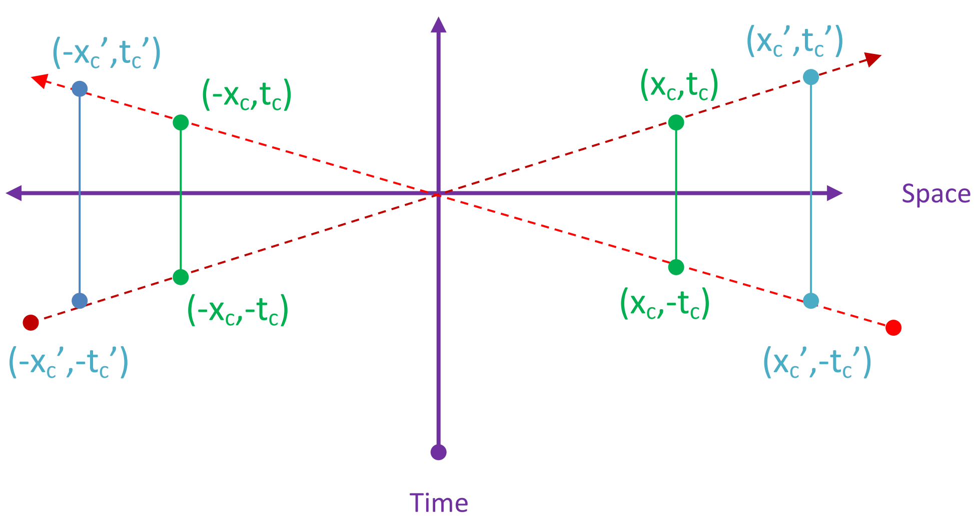
Part I

The Einstein/Bergmann derivation of the Lorentz Transform

I follow the derivation of the Lorentz Transform, following Peter S Bergmann in "Introduction to the Theory of Relativity" After an introductory explanation involving observers, trains, rods and clocks (in Cartesian coordinates), and with the assumption that the speed of light is a constant c, and v defines "inertial motion" relative to c, the actual analysis (pp. 38ff) begins with the hypothesis that the laws of physics are the same in every inertial frame. However, I interpret some of Bergmann's characterizations a bit differently.

The constancy of the speed of light implies there is an absolute coordinate frame in which c is defined as

$x_c = ct_c$ and $x_c = ct_c \Leftrightarrow x_c' = ct_c'$ and $c = \frac{|x_c|}{|t_c|} = \frac{|x'_c|}{|t'_c|}$



In this diagram, time moves up vertically, and the position coordinates of a photon modeled as a coordinate particle move right or left as the time axis crosses their red dotted "world lines". Since the photons do not interact at the point (0,0) shown, they continue their straight world lines after they coincide. For a pure coordinate system, it is immaterial "when" the coordinate photons are created or destroyed as long as their world lines exist at their common point (at that point, they are "simultaneous", at any other instant of time, they are separated.

The velocity of light is then defined as $c = \frac{|x_c|}{|t_c|}$ since c is assumed to be isotropic (independent of direction) and homogeneous. If a medium is assumed, then the velocity of the medium is then assumed to be defined as $v = \frac{x_v}{t_v}$ along some axis passing through the origin at which c is defined …

A Michaelson-Morley apparatus consists of two orthogonal and equal arms which were rotated to test the effect of a motion v in some direction with respect to c in terms of an interference pattern, which would change along the arms as the system was rotated. Since no effect was observed, Lorentz hypothesized that the arms of the apparatus had actually changed to compensate for the interaction of the medium with the apparatus.

Lorentz assumed that the actual length of the arm in the direction of velocity would be changed by a factor linear α that would vary as the system was rotated, causing a variation in space and time by interaction with the medium.

$x' = \alpha(x - vt),\ y' = y = 0,\ z' = z = 0$ (since y is orthogonal to this axis, and z is irrelevant)

In particular, for v = 0 (an orientation orthogonal to the relative velocity of the medium), $x' = \alpha x,\ v = 0$. For the orientation in line with the relative velocity of the medium, $x = vt$, so that $x' = 0$, with α to be determined. In the direction orthogonal to the position characterized by α he then assumed that the measurement of time would be affected by a linear factor of both space and time, and replacing "y" with x':

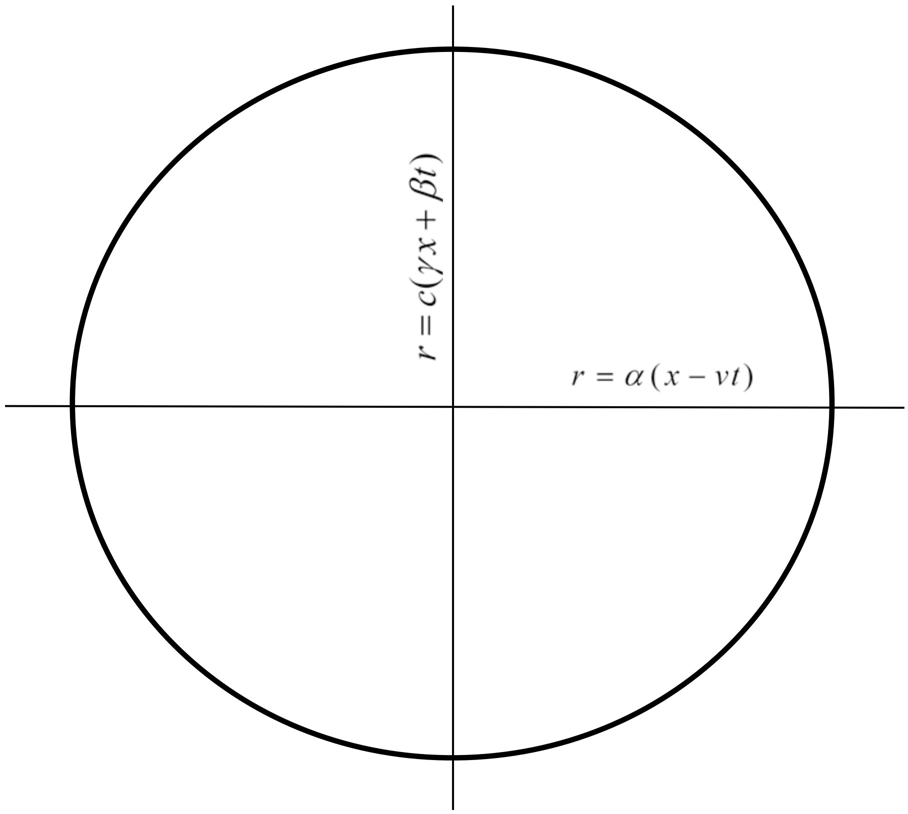
$t' = \gamma x + \beta t$ so that

$x' = ct' = c(\gamma x + \beta t)$

Equating the two characterizations of x', we have the relation:

$\alpha(x - vt) = c(\gamma x + \beta t)$

Since these results apply to arms that are orthogonal, there are no interacting terms, and the apparatus is described by the lengths of the rotating arms in a medium where c is homogeneous and isotropic, and the arm length change as the system is rotated:



The area of the circle can then be calculated by the relation:

$$\pi\alpha^2(x - vt)^2 = \pi c^2(\gamma x - \beta t)^2$$

Eliminating π and expanding and rearranging results in the expression:

$$(c^2\beta^2 - v^2\alpha^2)t^2 = (\alpha^2 - c^2\gamma^2)x^2 + (v\alpha^2 + c^2\beta\gamma)xt$$

Equating the coefficients of x^2 and t^2 (c is a constant independent of area where $s_x = x^2 = xx$ and $s_t = t^2 = tt$) ,and setting the coefficient of xt equal to 0 (no interaction between space and time – i.e., x and t are orthogonal; this is equivalent to using dot and cross products in the integral form of Maxwell's equations, and the elimination of the scalar field to form the EM field tensor in Quantum Electrodynamics) results in the expressions:

$$1:\ c^2\beta^2 - v^2\alpha^2 = c^2$$
$$2:\ \alpha^2 - c^2\gamma^2 = 1$$
$$3:\ v\alpha^2 + c^2\beta\gamma = 0$$

Equating α^2 from 1,2:

$$1:\ \alpha^2 = \frac{c^2(1-\beta^2)}{v^2}$$
$$2:\ \alpha^2 = 1 + c^2\gamma^2$$

$$\frac{c^2(1-\beta^2)}{v^2} = 1 + c^2\gamma^2$$
$$4:\ \gamma^2 = \frac{1}{c^2}\left[\frac{c^2(1-\beta^2)}{v^2} - 1\right]$$

Equating α^2 from (2,3)

$$2:\quad \alpha^2 = 1 + c^2\gamma^2$$
$$3:\quad \alpha^2 = -\frac{c^2\beta\gamma}{v}$$
$$2,3:\ 1 + c^2\gamma^2 = -\frac{c^2\beta\gamma}{v}$$

From 1,3

$$\frac{c^2(1-\beta^2)}{v^2} = -\frac{c^2\beta(\gamma)}{v}$$
$$5:\ \gamma = -\frac{(1-\beta^2)}{v\beta}$$

Squaring 5 and equating with 4:

$$\left(-\frac{(1-\beta^2)}{v\beta}\right)^2 = \frac{1}{c^2}\left[\frac{c^2(1-\beta^2)}{v^2} - 1\right]$$

After a bit of algebra, this results in the expression:

$$\beta^2 = \frac{1}{1-\frac{v^2}{c^2}}$$

Substituting in 5:

$$5:\gamma = -\frac{(1-\beta^2)}{v\beta} = -\frac{1}{v}\left[\frac{(1-\beta^2)}{\beta}\right] = -\frac{1}{v}\left[\frac{(1-\frac{1}{1-\frac{v^2}{c^2}})}{\beta}\right]$$

$$1-\frac{1}{1-\frac{v^2}{c^2}} = 1-\frac{c^2}{c^2-v^2} = \frac{c^2-v^2-c^2}{c^2-v^2} = \frac{-v^2}{c^2-v^2}$$

$$\gamma = -\frac{1}{v}\frac{\frac{v^2}{c^2-v^2}}{\beta} = \frac{1}{v}\frac{\frac{v^2}{(c+v)(c-v)}}{\beta} = -\frac{\frac{v}{(c+v)(c-v)}}{\sqrt{\frac{1}{1-\frac{v^2}{c^2}}}} = -\frac{v\sqrt{1-\frac{v^2}{c^2}}}{(c+v)(c-v)} =$$

$$-\frac{v\sqrt{\frac{c^2-v^2}{c^2}}}{(c+v)(c-v)} = \frac{\frac{v}{c}\sqrt{c^2-v^2}}{(c+v)(c-v)} = -\frac{\frac{v}{c}\sqrt{(c+v)(c-v)}}{(c+v)(c-v)} = -\frac{\frac{v}{c}}{\sqrt{(c+v)(c-v)}} =$$

$$-\frac{\frac{v}{c}}{\sqrt{c^2-v^2}} = -\frac{\frac{v}{c}}{c\sqrt{1-\frac{v^2}{c^2}}} = \frac{v}{c^2}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = -\frac{v\beta}{c^2}$$

That is, $\gamma = \frac{1-\beta^2}{v\beta} = -\frac{\beta v}{c^2}$

Substituting in Equation 3:

$$3: \quad \alpha^2 = -\frac{c^2\beta\gamma}{v} = \left(-\frac{c^2\beta}{v}\right)\left(-\frac{\beta v}{c^2}\right) = \beta^2$$

These expressions for α , β and γ are then substituted into the original equations

$$x' = \alpha(x - vt) = \beta(x - vt)$$

$$t' = \gamma x + \beta t = -\frac{\beta v}{c^2} x + \beta t = \beta\left[t - \left(\frac{vx}{c^2}\right)\right]$$

$$\beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} =$$

We have the Lorentz transforms (from Bergmann):

$$x' = \frac{(x - vt)}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t' = \frac{t - \left(\frac{vx}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$$

Important Note:

Bergmann's parameters γ and β are reversed from that of conventional notation in my experience, and in the following analyses the notation will be reversed as well.

The result is the Lorentz transform:

$$x' = (x - vt)\gamma$$
$$z' = z \ \ (=0)$$
$$t' = (t - \frac{vx}{c^2})\gamma$$
$$with\ \gamma = \frac{1}{\sqrt{1-\beta^2}}, \beta = \frac{v}{c}$$

(where t has replaced y in a spatial conception of the model)

The transform in (x,t) is:

$$x' = (x - vt)\gamma$$
$$t' = (t - \frac{vx}{c^2})\gamma$$
$$with\ \gamma = \frac{1}{\sqrt{1-\beta^2}}, \beta = \frac{v}{c}$$

The "Time Dilation" Equation

Note that for x = ct,

$$t' = (t - \frac{vx}{c^2})\gamma$$
$$x' = ct' = (ct - \frac{vx}{c})\gamma = (ct - vt)\gamma = (x - vt)\gamma$$

That is, from a purely algebraic perspective, under the condition x=ct, the time and space transforms are identical. This is because v has not been specified for both positive and negative values ("parity"), which means we must also consider the relation $x' = (x + vt)\gamma$

Adding these two together gives: $2x' = 2x\gamma$ so that $x' = x\gamma$

For x = ct and x'=ct', this gives the so-called "time dilation" equation (notice that x has been explicitly eliminated by the relations t = x/c and t'=x'/c:

$$t' = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}} = t\gamma$$

“Time” can then be characterized as a then a scaling factor that describes a velocity v in relation to a velocity c for a given length or radius that relates v to c, with all parameters independent of whether a Galilean or a Radial coordinate system is selected.

I indicate this by using capital letters for these parameters, so that:

$$T' / T = \frac{1}{\sqrt{1 - \frac{V^2}{C^2}}}$$

Time is then said to be covariant with velocity for either coordinate system, since it transforms in direct proportion to velocity.

Part II - The Covariant formulation of STR

The relativistic Equations of Special Theory of Relativity can be derived independently of a coordinate system, where C and V are characterized as mass creation rates and T and T' are considered as creation times, with a change in the total mass of the (single) system from M_0=CT to M'=CT' is characterized by a perturbation VT' where V is defined in terms of C by the relation $\beta = \frac{V}{C}$.

We assume that C is derived from Coulomb's and Ampere's laws by Maxell's equations, so that

$$C^2 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

If perturbation V is independent of C, so that the system mass before and after the perturbation can be characterized by

$$(CT')^2 = (VT')^2 + (CT)^2 \text{ or}$$

$$(M')^2 = (VT')^2 + (M_0)^2 = (M_V)^2 + (M_0)^2$$

(M_V is the mass added by during the perturbation T' at the creation rate V)

$$(M')^2 \mathrm{C}^4 = (VT')^2 \mathrm{C}^4 + (M_0)^2 \mathrm{C}^4$$

$(VT')^2 \mathrm{C}^4 = (VCT')^2 \mathrm{C}^2 = (M'V)^2 C^2 = P^2 C^2$, where $P = M'V$ is the relativistic momentum.

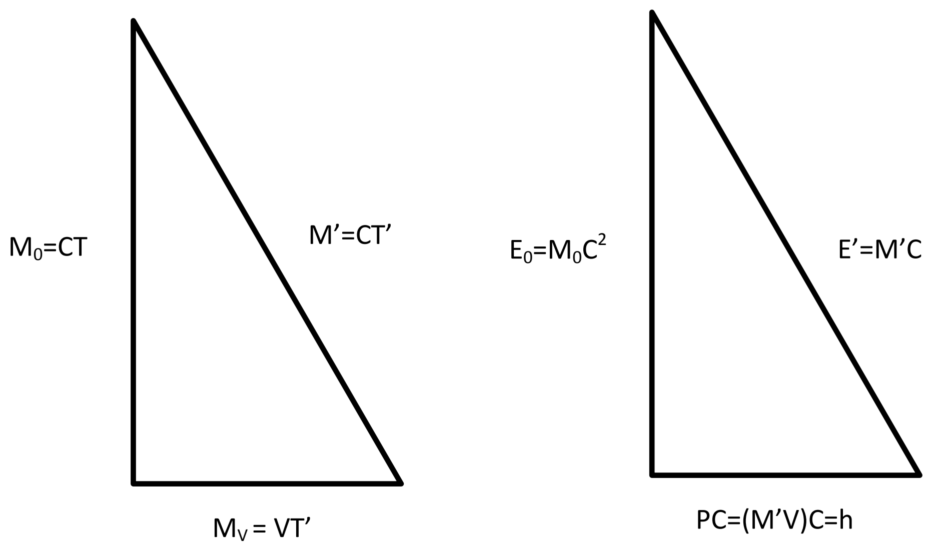
Then the relativistic energy equation is

$$(M'C^2)^2 = P^2 \mathrm{C}^2 + (M_0 \mathrm{C}^2)^2$$
$$E'^2 = P^2 C^2 + E_0^{\,2}$$

The ratio of mass creation times is given by;

$$\frac{T'}{T_0} = \frac{1}{\sqrt{1 - \frac{V^2}{C^2}}}, \quad that\, is\ T' = \frac{T_0}{\sqrt{1 - \frac{V^2}{C^2}}}$$

The following diagrams show these relationships:



I have included h as a precursor to relativistic quantum mechanics. Here the interpretation is that if there is a change of state (from E' to E_0) in a distant source emitting an photon of energy h that Is detected locally, and there is a change in the sensor of the same amount, then nothing has interacted with it along its path. However, if there is a change due to interaction, the observed energy of the photon will decrease, which is observed as a change in h, and is responsible for the red shift.

Note that Doppler (first or second order) is not an issue here, since the coherence length of the photon is tiny compared to the length of the journey.

Part III Co-Variance and Contra-Variance

We have seen that the Lorentz transform can be interpreted as a "time dilation" equation in a space-time coordinate system (x,t) with the assumption of the constant velocity of light c if both positive and negative velocities are included in the analysis (which satisfies the condition of isotropy of both v and c for physical laws), where all other velocities are determined by the relation v/c.

We have also seen that the same equation can be derived independently of a coordinate system in the Energy-Momentum system (E,K) for a single particle (system) with an initial M_0 perturbed by an additional mass M_V to a final mass M', with the transition described by scaling factors (T,T')

The two domains (x,t) and (E,K) are independent of each other; the Lorentz transformation was accomplished without reference to mass, and the Mass-Energy relationships were derived without reference to a coordinate system.

If we set the ratios v/c = V/C with c=C and v = V we can then relate the (E,K) domain to the (x,t) domain by breaking v and c into their space-time components.

The ratio of velocities in Galilean coordinates is: $\frac{v}{c}=\frac{\left(\frac{\Delta x_v}{\Delta t_v}\right)}{\left(\frac{\Delta x_c}{\Delta t_c}\right)}=\frac{\Delta x_v}{\Delta x_c}\frac{\Delta t_c}{\Delta t_v}$, where Δ indicates that discrete lengths and time intervals are defined.

In the relativistic (E,K) domain, the mass creation rates V and C are related by the scaling factors ΔT and ΔT' by:

$$V=C\sqrt{1-\left(\frac{\Delta T_0}{\Delta T'}\right)^2}\ ,\ \Delta T'\geq\Delta T_0$$

$$\frac{v}{c}=\frac{\Delta x_v}{\Delta x_c}\frac{\Delta t_c}{\Delta t_v}=\frac{V}{C}=\sqrt{1-\left(\frac{\Delta T_0}{\Delta T'}\right)^2}=\sqrt{1-\left(\frac{\Delta CT_0}{\Delta CT'}\right)^2}=\sqrt{1-\left(\frac{\Delta M_0}{\Delta M'}\right)^2}$$

For a change in v/c, the relation c=C can be preserved for variations in energy provided that one of the parameters C and c have a coordinate in common. That means there are two possible solutions; that in which a common spatial interval is assumed, and that in which a common interval of time is assumed.

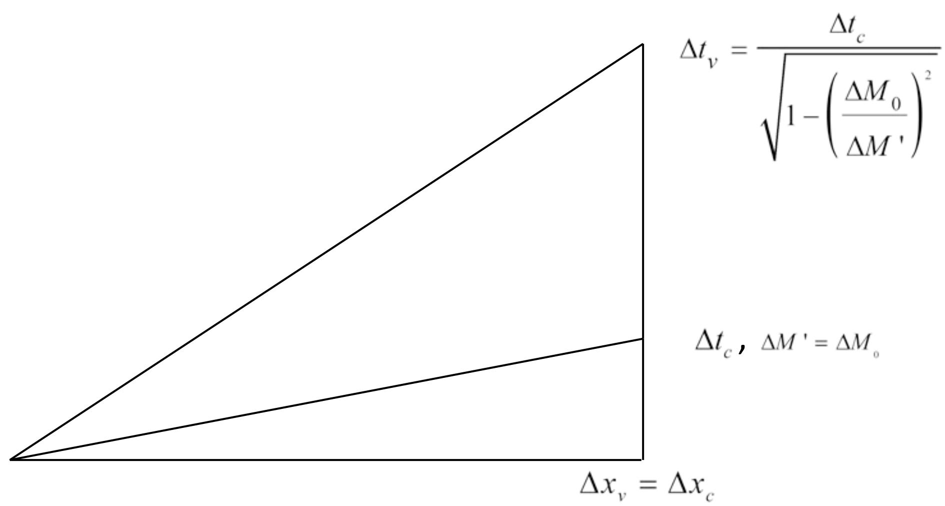
Case 1: $\Delta x_v = \Delta x_c$ (**co-variant** transformation)

In this case, we have $\frac{v}{c} = \frac{\Delta t_c}{\Delta t_v} = \sqrt{1 - \left(\frac{\Delta M_0}{\Delta M'}\right)^2}$, so that

$$\Delta t_v = \frac{\Delta t_c}{\sqrt{1 - \left(\frac{\Delta M_0}{\Delta M'}\right)^2}}$$

For M' = M_0, we have that $\Delta t_v = \Delta t_c$ As M' increases along with V, Δt_v also increases. Since M' in (E,K) is directly proportional to Δt_v in the coordinate system, we say that the transformation is **co-variant**.

(We have ignored parity, since we assume equal values set at the midpoint by squaring the parameters in the solution).



The interpretation of covariance is that an inertial object (for v < c) will take longer to traverse a given distance than a photon, with an increased total mass determined by the ratio M_0/M' .

Here the factor $\frac{1}{\sqrt{1 - \left(\frac{\Delta M_0}{\Delta M'}\right)^2}}$ is interpreted as an increased density of the system as a whole for V/C.

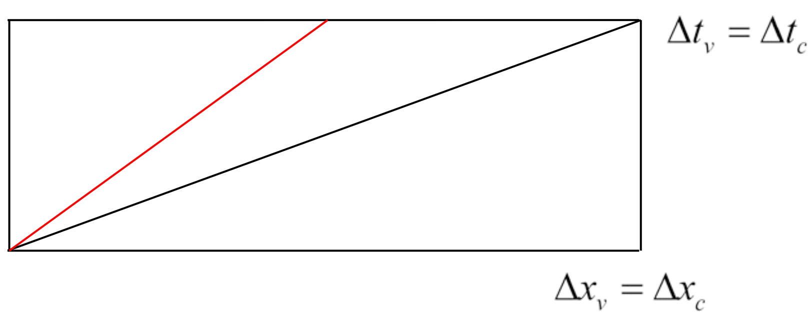
Case 2: $t_v = t_c$ (**contra-variant** transformation)

In this case, we have $\frac{v}{c} = \frac{\Delta x_v}{\Delta x_c} = \frac{V}{C} = \sqrt{1 - \left(\frac{\Delta M_0}{\Delta M'}\right)^2}$, so that

$$\Delta x_v = \Delta x_c \sqrt{1 - \left(\frac{\Delta M_0}{\Delta M'}\right)^2}$$

For $\Delta M' = \Delta M_0$, we have $\Delta x_v = 0$, which means that no additional mass has been added to the rest mass. As $\Delta M'$ increases, Δx_v decreases, meaning that an inertial object for v < c will not travel as far in a given time as a photon. Again, the factor $\sqrt{1 - \left(\frac{\Delta M_0}{\Delta M'}\right)^2}$ can be interpreted as a density, which increases as increasing mass (relative to that of a photon); however, since the result is inversely proportional to V/C, the transformation is **contra-variant** with respect to the coordinate system.

$$\Delta x_v = \Delta x_c \sqrt{1 - \left(\frac{\Delta M_0}{\Delta M'}\right)^2} = \Delta x_c \sqrt{1 - \left(\frac{\Delta T_0}{\Delta T'}\right)^2}$$



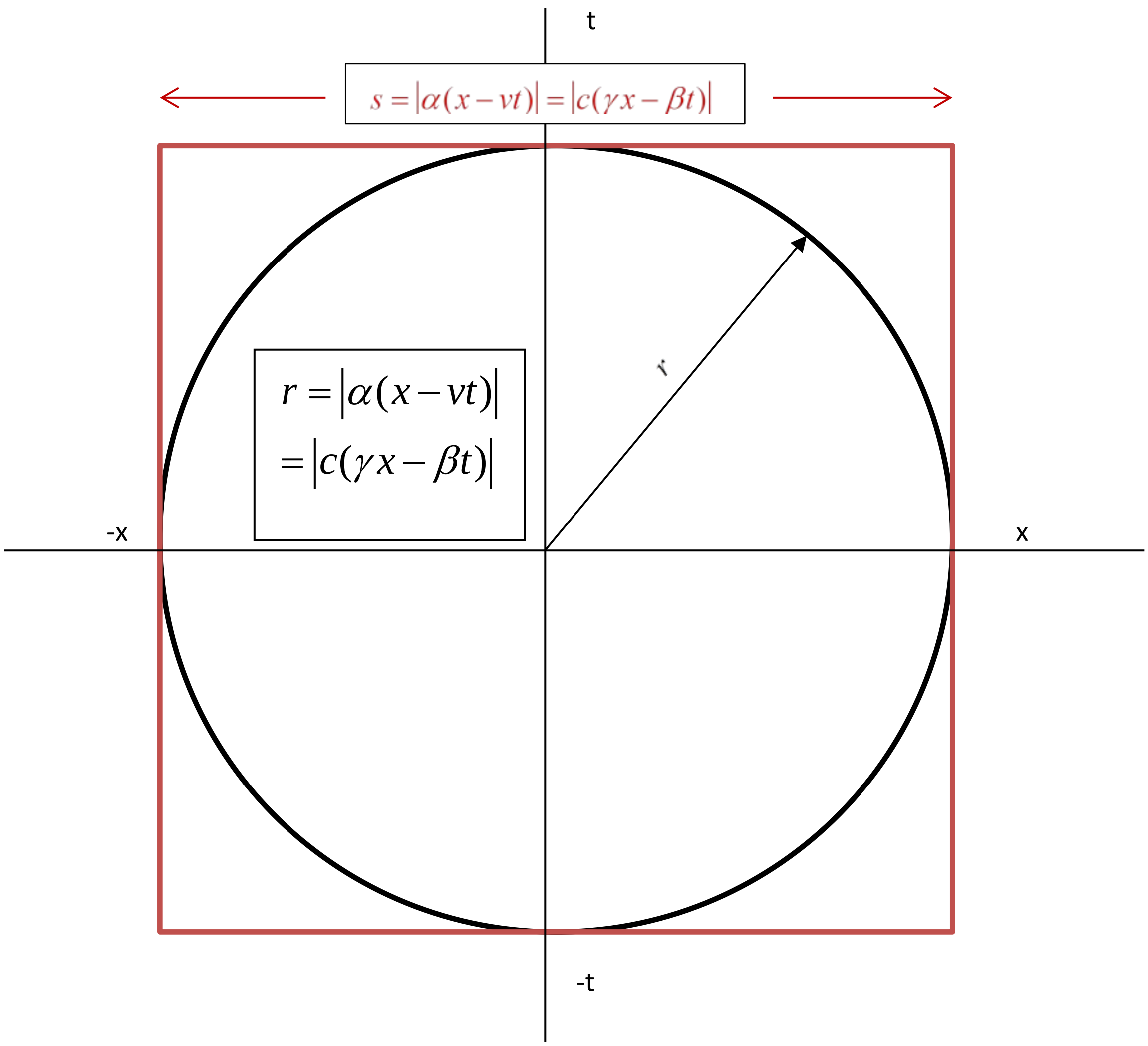
The Cartesian vs. Radial coordinates

At this point in the analysis, it is evident that the context is that of an area; either that of a square where each element on the left and right side are interpreted as its sides, or as a circle, in which each side is interpreted as a radius, and:

Area$_{Square}$ = $s^2 = \alpha^2(x - vt)^2 = c^2(\gamma x - \beta t)^2$

Area$_{Circle}$ = $r^2 = \pi\alpha^2(x - vt)^2 = \pi[c^2(\gamma x - \beta t)^2]$

The two are, of course, related by π , since



The analysis of the Lorentz transform in Cartesian coordinates violates the assumption of the homogeneity of space and time, since the distance from the origin at (0,0) to the perimeter of the square is only equal to that of the circle at the horizontal and vertical axes of the diagram above (t = t' = 0 or x' = x = 0).

This means that v is only independent of c at these points. However, if c is a constant, then v must be independent of c for all values of v, which can only be true if the relation $x' = \alpha(x - vt), y' = y, z' = z$ is taken to be that of a radius, in which case v is orthogonal to c as a tangent to the circumference.

$$\begin{aligned} r' &= \alpha(r - vt), \\ \theta' &= \theta = 2\pi, \\ \varphi' &= \varphi = 2\pi \\ t' &= \gamma r - \beta t \end{aligned}$$

and

$$r = ct \Leftrightarrow r' = ct'$$

The same analysis fan be performed, with the result that

$$r' \;=\; ct' \;=\; \frac{(r - vt)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Also, note that v is no longer defined by the relation $v \;=\; x/t$ but rather by the relation:

$$v = c\sqrt{1 - \frac{t^2}{t'^2}} = c\sqrt{1 - \frac{(ct)^2}{(ct')^2}} = c\sqrt{1 - \frac{r^2}{r'^2}},\; t' \geq t$$

If v= 0 the denominator (or equivalently, positive and negative radii are included), then

$$r' \;=\; \frac{r}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and r is co-variant with velocity, as are mass and energy. However the density at a point on the circumference is contra-variant with r.