Quaternions

Charles Keyser

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Quaternions

$$\vec{i}\otimes\left(\vec{k}\otimes\vec{j}\right)\triangleq ikj=1$$
$$\vec{i}\otimes\left(\vec{j}\otimes\vec{k}\right)\triangleq ijk=-1$$

$$\left(\vec{i}\otimes\vec{j}\right)=\vec{k}$$
$$\left(\vec{j}\otimes\vec{i}\right)=-\vec{k}=\overleftarrow{k}$$
$$\left(\vec{i}\otimes\vec{j}\right)+\left(\vec{i}\otimes\vec{j}\right)=\vec{k}-\vec{k}=\vec{k}+\overleftarrow{k}=\vec{0}$$

$$\vec{i}\otimes\left(\vec{j}\otimes\vec{k}\right)$$
$$\left(\vec{j}\otimes\vec{k}\right)=\vec{l}$$
$$\left(\vec{k}\otimes\vec{j}\right)=-\vec{l}=\overleftarrow{l}$$
$$\left(\vec{j}\otimes\vec{k}\right)+\left(\vec{k}\otimes\vec{j}\right)=\vec{l}-\vec{l}=\vec{l}+\overleftarrow{l}=\vec{0}$$

$$\left(\psi_i\right)=r_i+\vec{i}\otimes\left(\vec{j}\otimes\vec{k}\right)$$
$$\left(\psi_i\right)^*=r_i-\vec{i}\otimes\left(\vec{k}\otimes\vec{j}\right)$$
$$\left(\psi_i\right)\left(\psi_i\right)^*=\left(r_i\right)^2+r_i\left(\vec{i}\otimes\left(\vec{j}\otimes\vec{k}\right)\right)-r_i\left(\vec{i}\otimes\left(\vec{j}\otimes\vec{k}\right)\right)-\left(r_i\right)^2\left[i\otimes\left(\vec{j}\otimes\vec{k}\right)\right]^2=\left(r_i\right)^2-\left[\vec{i}\otimes\left(\vec{j}\otimes\vec{k}\right)\right]^2$$
$$\left(R_i\right)^2\triangleq\left(\psi_i\right)\left(\psi_i\right)^*=\left(r_i\right)^2-\left[\vec{i}\otimes\left(\vec{j}\otimes\vec{k}\right)\right]^2$$
$$\left[\left(R_i\right)^2\right]^*\triangleq\left[\left(\psi_i\right)\left(\psi_i\right)^*\right]^*=\left(r_i\right)^2-\left[\vec{i}\otimes\left(\vec{j}\otimes\vec{k}\right)\right]^2$$
$$\left[\left(R_i\right)^2\right]\left[\left(R_i\right)^2\right]^*=\left\{\left(r_i\right)^2+\left[\vec{i}\otimes\left(\vec{j}\otimes\vec{k}\right)\right]^2\right\}\left\{\left(r_i\right)^2-\left[\vec{i}\otimes\left(\vec{j}\otimes\vec{k}\right)\right]^2\right\}=\left[\left(r_i\right)^2\right]^2\left\{\left[\vec{i}\otimes\left(\vec{j}\otimes\vec{k}\right)\right]^2\right\}^2$$
$$=\left[\left(r_i\right)^2\right]^2\left\{\left[\vec{i}\otimes\vec{l}\right]^2\right\}^2$$

$$\left[\left(r_i\right)^2\right]^2\left\{\left[\vec{i}\otimes\vec{l}\right]^2\right\}^2$$

$$\vec{i}\perp\vec{l}\Rightarrow\left[\vec{i}\otimes\vec{l}\right]=0\Rightarrow\cos\left(\theta_{il}\right)=0$$

$$\Rightarrow\left(R_i\right)^2=\left(\psi_i\right)\left(\psi_i\right)^*=\left(r_i\right)^2$$

$$\left(R_{ijk}\right)^2=\left(r_i\right)^2+\left(r_j\right)^2+\left(r_j\right)^2$$

Quaternions eliminate the interaction between the imaginary axes in 3-space, leaving only the real radii. For vectors then

$$\begin{vmatrix} r & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & k \end{vmatrix}\otimes\begin{vmatrix} r & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & k \end{vmatrix}=\left(R_{ijk}\right)^2=\left(r_i\right)^2+\left(r_j\right)^2+\left(r_j\right)^2\Leftrightarrow\text{r real (positive definite), }\vec{i},\vec{j},\vec{k}\text{ imaginary}$$

$$\vec{i}\triangleq\left(\sqrt{-1}\right)\vec{i},\ \vec{j}\triangleq\left(\sqrt{-1}\right)\vec{j},\ \vec{k}\triangleq\left(\sqrt{-1}\right)\vec{k}$$

$$\pi\left(R_{ijk}\right)^2=\pi\left(r_i\right)^2+\pi\left(r_j\right)^2+\pi\left(r_j\right)^2=\pi\left(\left(r_i\right)^2+\left(r_j\right)^2+\left(r_j\right)^2\right)$$