

Coordinate Transformations (Variance)

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[Web Site](#)

[Physics Page](#)

References

[Proof of Fermat’s Theorem \(and its relation to Relativity\)](#)

[Proof of Goldbach’s Conjecture](#)

[Contravariance and Covariance](#) (Wikipedia)

(This document is undergoing a major revision for clarity. Stay tuned -will be back in a day or so.

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Covariant and Contravariant Transformations

Preliminary Assumptions

Assume that the characterization of positive numbers is the condition of existence, so that all references to variables in the following section are positive definite, including $\sin \theta$ and $\cos \theta$ which will be defined in a following section (for now, just consider them as variables of positive real numbers under the addition operation); the conditions on θ will be introduced in a subsequent section ([Sine and Cosine Transformation](#)).

Covariant Transformation

A **covariant** transformation transforms a unit basis vector by multiplication of a real number

$$\begin{aligned}([T_x]x)\vec{i} &= x\vec{i} = (x_i)\vec{e}^i \\ ([T_x]x)\vec{i} \cdot ([T_x]x)\vec{i} &= x_i(\vec{e}^i) \cdot x_i(\vec{e}^i) = x_i^2(\vec{e}^i \cdot \vec{e}^i) = x_i^2(\vec{i} \cdot \vec{i}) = x_i^2 \\ ([T_x]\sqrt{x})\vec{i} \cdot ([T_x]\sqrt{x})\vec{i} &= [(\sqrt{x})]\vec{i} \cdot [(\sqrt{x})]\vec{i} = x(\vec{i} \cdot \vec{i}) = x\end{aligned}$$

where x is not a ratio of numbers.

Contravariant Transformation

A contravariant transformation is defined by $[T^x(x)]\vec{i} \triangleq (x')\vec{i} \triangleq x' \vec{e}_i$ on a ratio where

$$[T^x(x)]\vec{i} = \left\{ (x')\vec{i} \cdot \left[\frac{1}{x} \right] \vec{i} \right\} \vec{i} = \left\{ (x')\vec{e}_i \cdot \left[\frac{1}{x} \right] \vec{e}_i \right\} \vec{i} = \left\{ \left(\frac{x'}{x} \right) (\vec{i} \cdot \vec{i}) \right\} \vec{i} = \left\{ \left(\frac{x'}{x} \right) (\vec{e}_i \cdot \vec{e}_i) \right\} \vec{i} = \left(\frac{x'}{x} \right) \vec{i}$$

Discussion

Consider the equation

$$1 = \cos \theta + \sin \theta \quad (1.1)$$

In one dimension, the basis is given by the unit vector \vec{i} , so that

$(1)\vec{I} = (1)[\cos \theta + \sin \theta]\vec{I}$, where $(1)[\cos \theta + \sin \theta]$ is called the **coefficient** of the basis vector \vec{i} and (1) is called the **amplitude** of the coefficient function. In order that (1) be invariant, the relation (1.1) must be valid for all values of θ . The coefficients of vectors are also called **scalars**, since they “scale” their associated vectors.

If (1.1) is multiplied by a constant that changes the amplitude so that

$$(v)\vec{I} = (v)[\cos \theta + \sin \theta]\vec{I}, \quad (1.2)$$

Then (v) can be considered a transformation on the unit vector \vec{i} so that

$$v = T_v(1) = \left[\sqrt{T_v(1)} \right] \vec{i} \cdot \left[\sqrt{T_v(1)} \right] \vec{i} = T_v(1) (\vec{i} \cdot \vec{i}) \quad \text{so that } T_v(1) = v \vec{i} = \vec{1}_v \quad \text{as a transformed unit basis.}$$

Then

$$1 = T_\theta(1) = \left[\sqrt{T_\theta(1)} \right] \vec{i} \cdot \left[\sqrt{T_\theta(1)} \right] \vec{i} = \left[\sqrt{(\cos \theta + \sin \theta)} \right] \vec{i} \cdot \left[\sqrt{(\cos \theta + \sin \theta)} \right] \vec{i} = (\cos \theta + \sin \theta) (\vec{i} \cdot \vec{i})$$

So that $T_\theta(1) = 1 = \cos \theta + \sin \theta$ is also a transformation on the amplitude of the basis vector $\vec{1}$ to a new basis $T_\theta(1) \vec{i} = (\cos \theta + \sin \theta) \vec{i} = \vec{i}_\theta$.

Similarly, $T_v \vec{i} = v(1) \vec{i} = v \vec{i}$ is a transformation on the amplitude (1) . Since both transformations are in the same dimension, the product

$$v = T_v \cdot T_\theta = \left[T(v(1)) \right] \vec{i} \cdot \left[T(\cos \theta + \sin \theta) \right] \vec{i} = \left[v \right] \vec{i} \cdot \left[(\cos \theta + \sin \theta) \right] \vec{i} = \left[v(\cos \theta + \sin \theta) \right] (\vec{i} \cdot \vec{i}) = v(\cos \theta + \sin \theta)$$

for $1 = (\cos \theta + \sin \theta)$, where both transformations can be considered transformations on the same basis vector \vec{i} .

Then v changes with the transformation $v = \left[\sqrt{T_v(1)} \right] \vec{i} \cdot \left[\sqrt{T_v(1)} \right] \vec{i} = \left[\sqrt{v} \right] \vec{i} \cdot \left[\sqrt{v} \right] \vec{i} = v (\vec{i} \cdot \vec{i})$

provided the basis $\left[T_\theta(1) \right] \cdot \left[T_\theta(1) \right] = \left[(1) \vec{i} \right] \cdot \left[(1) \vec{i} \right] = \left(\sqrt{(\cos \theta + \sin \theta)} \right) \vec{i} \cdot \left(\sqrt{(\cos \theta + \sin \theta)} \right) \vec{i}$,

$$\text{so that } \left[T_\theta \right] \cdot \left[T_\theta \right] = (\cos \theta + \sin \theta) (\vec{i} \cdot \vec{i}) = (\cos \theta + \sin \theta) = 1$$

is invariant, the transformation $T_v(1) = v$ is called **covariant** and the transformation

$T_\theta(1) = (\cos \theta + \sin \theta) = 1$ is called **invariant**.

If $v = 1$ then $\left[T_v \right] \cdot \left[T_\theta \right] = v = 1$, so that $T_\theta = \frac{1}{T_v}$, and vice versa, then the transformations are

reciprocal, and are called **contravariant**.

Summary

If there is a product transformation $[T_v] \cdot [T_\theta]$ in a single dimension, then

1. If the relationship between two transformations consists of a **variant** (e.g.) T_v and an **invariant** (e.g. T_θ), then the product transformation is called **covariant**, and

$$v' = [T_v(1)] \vec{i} \cdot [T_\theta(1)] \vec{i} = v'(\cos \theta + \sin \theta) = v'(1) \text{ changes with the variant.}$$

2. If both transformations are **variant**

$$v' = v([T_v] \vec{i} \cdot [T_\theta] \vec{i}) = ([v]) \vec{i} \cdot (\cos \theta + \sin \theta) \vec{i} = v(\cos \theta + \sin \theta) (\vec{i} \cdot \vec{i}) = v(\cos \theta + \sin \theta)$$

are variant, then v' changes with both variants.

3. If neither transformation is variant, then $v = v([T_v] \cdot [T_\theta]) = v([\sqrt{(1)}] \vec{i} \cdot [\sqrt{(1)}] \vec{i}) = v(\vec{i} \cdot \vec{i})$

then v does not change.

4. If both transformations change, but the product transformation is invariant, then the transformations are reciprocal, and the product transformation is called **contravariant**, where.

$$\text{and } v' = [T_v] \vec{i} \cdot \left[\frac{1}{T_\theta} \right] \vec{i} = \left[\frac{T_v}{T_\theta} \right] (\vec{i} \cdot \vec{i}) = \frac{v}{(\cos \theta + \sin \theta)} = \frac{v}{(\cos \theta + \sin \theta)}, \text{ where } v \text{ and}$$

$(\cos \theta + \sin \theta)$ are inversely proportional. Note that if both variant transformations are

multiplied by the same scalar t , then $tT_\theta = tT_v \Leftrightarrow T_\theta = T_v$, so the product transformation remains invariant.

Note that the relative change of basis is the same in one dimension if the covariant and contravariant transforms are the same; i.e. there is no distinction between the transformations in terms of the basis vectors) since for

$$[v'] \vec{i} = \left[\frac{v}{(\cos \theta + \sin \theta)} \right] \vec{i} \text{ is valid if and only if } [v'] [(\cos \theta + \sin \theta)] \vec{i} = [v] \vec{i} \text{ so that all such}$$

transformations of the product two coefficients can be cast as either covariant or contravariant for

$$(1) \vec{i} = \frac{[v'] [(\cos \theta + \sin \theta)]}{[v'] [(\cos \theta + \sin \theta)]} = \frac{[v] [(\cos \theta + \sin \theta)]}{[v] [(\cos \theta + \sin \theta)]}.$$

Example (deBroglie)

Consider the deBroglie characterization of the relativistic energy of the photo-electric effect:

$E = (\sqrt{E})\vec{i} \cdot (\sqrt{E})\vec{i} = E(\vec{i} \cdot \vec{i}) = [h]\vec{i} \cdot [\nu]\vec{i} = [h\nu](\vec{i} \cdot \vec{i}) = h\nu$., where h is also an energy (the “action”).

1. If E will varies with both h and ν as variants, then E , h , and ν are all **covariant** transformations on the unit basis vector \vec{i} , and will vary with the transformation product $h\nu$
2. If E does not vary at all, then both h and ν are **invariant**.
3. (**deBroglie**) If E varies with ν but not with h , then E and ν are variants, but h is **invariant**.
4. Consider the decomposition

$E = h\nu = h\left(\frac{c\tau}{\lambda}\right)$ where h is an invariant and $\nu = \frac{c\tau}{\lambda}$ Then if E is invariant, so is ν , and

therefore $\nu = \nu\left(\frac{c\tau}{\lambda}\right) = (c\tau)\vec{i} \cdot \left(\frac{1}{\lambda}\right)\vec{i} = \nu(\vec{i} \cdot \vec{i})$, where $\nu = (\sqrt{\nu})\vec{i} \cdot (\sqrt{\nu})\vec{i} = \nu(\vec{i} \cdot \vec{i})$. Then

$\frac{c\tau}{\lambda} = 1$ so the transformations $c\tau$ and λ on the unit vector are **contravariant**, and if either one of them changes so that E is variant, the relation between them is still contravariant – their transformations are reciprocal, or inversely proportional. Note that the both c and τ taken by themselves are both **covariant** so the product $c\tau$ is also covariant.

Therefore, a contravariant transformation is characterized by a ratio (e.g velocity) $\nu = \frac{x}{t}$, where the

ratio $\frac{x}{t}$ remains the same if both numerator and denominator are multiplied by the same constant

(scaled the same way), so that $\nu = \frac{kx}{kt} = \frac{x}{t}$. Then the transformation $\vec{\nu} = \left(\frac{x}{t}\right)\vec{i}$ is contravariant and

x and t are inversely proportional, so that $T_{\left(\frac{x}{t}\right)} = \left(\frac{x}{t}\right)\vec{i} \cdot \left(\frac{t}{x}\right)\vec{i} = \left(\frac{xt}{tx}\right)(\vec{i} \cdot \vec{i}) = 1$; note that the variables

x and t commute in the combined transformation $T_{\left(\frac{x}{t}\right)}$ which results in a scalar..

If v is multiplied by $\frac{1}{c}$ so that $\beta = \frac{v}{c} = v\vec{i} \cdot \left(\frac{1}{c}\right)\vec{i} = \left(\frac{v}{c}\right)(\vec{i} \cdot \vec{i})$ then $(\sqrt{\beta})\vec{i}$ is a covariant transformation on \vec{i} such that $\beta = (\sqrt{\beta})\vec{i} \cdot (\sqrt{\beta})\vec{i} = \beta(\vec{i} \cdot \vec{i})$. Note that the transformation $(\beta c)\vec{i} = v\vec{i}$ is covariant, that the transformation

$$\beta = \left(\frac{v}{c}\right) = v\vec{i} \cdot \left(\frac{1}{c}\right)\vec{i} = \left(\frac{v}{c}\right)(\vec{i} \cdot \vec{i}) \text{ is contravariant.}$$

but in the transformation $v\vec{i} = \begin{pmatrix} x \\ t \end{pmatrix}\vec{i}$ v is contravariant in the variables x and t , and $(vt)\vec{i} = x\vec{i}$ is covariant.

The Sine and Cosine Transformation

$$T_\theta(1)\vec{i} = (\cos \theta + \sin \theta)\vec{i} = \vec{i}_\theta$$