The Physics of Lie Groups and the Pauli Matrices

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Consider the relationships from the Binomial Expansion as a single valued function:

$$c = a + b$$
$$c^2 = (a+b)^2 = a^2 + b^2 + 2ab$$

(multiply by π for radial expression:

$\pi c^2 = \pi(a+b)^2 = \pi a^2 + \pi b^2 + a(2\pi b) = \pi a^2 + \pi b^2 + a(2\pi b)$ where the interaction is characterized by the radius a and the circumference b.

The term $a^2 + b^2$ expresses the existence of two particles at a field point that expresses Newton's law of equal and opposite force as energy, while the term $2ab = Det\begin{vmatrix} a & a \\ -b & b \end{vmatrix}$ characterizes the interaction between them

$$\left|a^2\right| \quad \begin{array}{c} -ab\rightarrow \\ \leftarrow ba- \end{array} \quad \left|b^2\right| \qquad \text{or} \qquad \begin{aligned} &\left|a^2\right|\,\left|-ab\rightarrow\right|\leftarrow ba-\ \left|b^2\right| \\ \equiv&\left|a^2\right|\,\left|-h\rightarrow\leftarrow h-\right|\ \left|b^2\right| \\ \equiv&\left|a^2\right|\rightarrow\left|h^2\right|\leftarrow\left|b^2\right| \end{aligned}$$

This can be expressed in matrix format as:

$c^2 = Tr\left|\mathrm{E}\right| + Det\left|\mathrm{H}\right| = Tr\begin{vmatrix} a^2 & 0 \\ 0 & b^2 \end{vmatrix} + Det\begin{vmatrix} a & b \\ -b & a \end{vmatrix}$. Where $\left|\mathrm{H}\right| \triangleq \begin{vmatrix} a & b \\ -b & a \end{vmatrix}$ is skew symmetric and $Det\left|\mathrm{H}\right| = 2ab$ represents the above interaction between the energies a^2 and b^2

Complex Conjugation

Note that the equation of a circle can only be generated from the first order relation $c = a + b$ by the complex conjugation:

$$c = a + ib$$
$$cc^* = (a+ib)(a-ib) = a^2 + b^2{}'$$

where the color magenta represents imaginary numbers in the complex plane. In particular, note that $c^2 \neq cc*$, where $cc* = a^2 + b^2$ characterizes the Pythagorean Theorem (and the equation of a circle).

Consider the interaction matrix $|\mathrm{H}| \triangleq \begin{vmatrix} b & a \\ -a & b \end{vmatrix}$ where the variables have been interchanged (since they commute). Setting $b=0$ results in the matrix $|a| \triangleq \begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix}$ where the $Tr|a| = 0$ (meaning the particle b does not exist) but $Det|a| = a^2$, which is equivalent to $Det\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} = a^2$

This is an expression of Fermat's last Theorem via the Binomial Expansion for the case $n=2$, since

$$\begin{aligned} &c^2 = (a+b)^2 = a^2 + b^2 + 2ab \\ &c^2 = a^2 + b^2 \Leftrightarrow ab = 0 \\ &ab \neq 0 \\ &c^2 \neq a^2 + b^2,\ QED \end{aligned}$$

This can than be extended for arbitrary values of n by induction (or comparison with the full Binomial Expansion)

$$\begin{aligned} &c^n = (a+b)^n = a^n + b^n + \mathrm{Re}\, m(a,b,n) \\ &c^n = a^n + b^n \Leftrightarrow \mathrm{Re}\, m(a,b,n) = 0 \\ &\mathrm{Re}\, m(a,b,n) \neq 0 \\ &c^n \neq a^n + b^n,\ QED \end{aligned}$$

(Imaginary numbers are complex only for those who think they are somehow real).

Interaction of Particle masses for n=2

Change in the Relativistic Unit Circle can be conceptualize as a change from a unit mass in the initial state to a unit mass in the final state (the change is in only one dimension, which does not interact with other dimensions, so geometry is irrelevant)

The first order expression of mass characterizes its existence as a positive definite value , and the second order definition (with no perturbation – ie. Squared) is an expression of equal and opposite self interacting mass/energy (Newton's equal and opposite force in his third law)

Interacting Particles/Fields n=2)

Self-interacting particles are represented in one dimension as

$|A| \Leftrightarrow |A|^2 = |A^2| \Leftrightarrow |A|^n = |A^n|$

$Tr(|A|) = A,\ Tr(|A|^n) = Tr(|A^n|) = A^n$, so the Trace is equal to the determinant

$Det|A| = A \Leftrightarrow Det|A^n| = A^n$

In two dimensions, interaction between two different particles is characterized by multiplication in the same dimension via the Relativistic Unit Circle where the particles can be parametrized by the variables

$a \triangleq c\tau,\ b \triangleq v\tau'$ Then $m_0 \triangleq c\tau,\ m_v \triangleq v\tau'$

Thus the change to the initial mass state has two components; the interacting particles' existence, is given by the single valued relationship characterizing existence of a single particle $m_0 = c\tau$ and a change given by an interacting particle/field $v\tau'$: $c\tau' = c\tau + v\tau'$. The energy of the system can the be characterized by the second order relation:

$(c\tau')^2 = (c\tau)^2 + (v\tau')^2 + 2(c\tau)(v\tau')$, where the interaction energy is $h^2 \triangleq 2(c\tau)(v\tau')$

In terms of the RUC, the second order existence change is given by $\tan\theta = (\gamma\beta)^2$, where $\gamma \triangleq \frac{\tau'}{\tau}$ and $\beta = \frac{v}{c}$; the first order change is given by by $\pm\frac{d(\tan\theta)}{d(\gamma\beta)} = \pm\frac{d(\gamma\beta)^2}{d(\gamma\beta)} = \pm 2(\gamma\beta) \triangleq \pm h^2$

Note that the unit vector is changed from an initial state to a final state (a "change of metric" in one "dimension"; that of mass/energy, and is complete in that dimension (no interaction with a second dimension.

There are broad consequences to this characterization, which is founded on the Binomial Expansionn as a single valued function for the case $n = 2$

Non-Interacting Particles

In this case one particle (the "rest" particle) is real, and the perturbing particle is imaginary.

Note that non-interacting particles cannot be represented by a single valued matrix repesenation except as the trace of a diagonal matrix; e.g.

$\begin{vmatrix} A & 0 \\ 0 & B \end{vmatrix} \Leftrightarrow \begin{vmatrix} A & 0 \\ 0 & B \end{vmatrix}^2 = \begin{vmatrix} A^2 & 0 \\ 0 & B^2 \end{vmatrix}$ In this case, the trace can be expressed, but not the determinant, since $\vec{A}$ and $\vec{B}$ are affine (without a common origin), and so do not interact.

$$|a^2| \quad \begin{matrix} -ab\rightarrow \\ \leftarrow ba- \end{matrix} \quad |b^2| \neq |A^2|,\ |B^2|,\ |\leftarrow BA-|,\ |-AB\rightarrow|$$

Parameterizing:

$|c\tau'| = \begin{vmatrix} c\tau & 0 \\ 0 & v\tau' \end{vmatrix}$ where $|c\tau'|^2 = \begin{vmatrix} c\tau & 0 \\ 0 & v\tau' \end{vmatrix}^2 = \begin{vmatrix} (c\tau)^2 & 0 \\ 0 & (v\tau')^2 \end{vmatrix}$ where $Tr|c\tau'| = c\tau + v\tau'$ characterizing existence, but $Det|c\tau'| = (c\tau)(v\tau')$ which is the "dot" product (multiplicative scalar in one dimension and $(c\tau)\otimes(v\tau') = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (c\tau)(v\tau') \end{vmatrix}$ which is the multiplicative scalar product in a third non-interacting dimension.

Note that for these non-interacting particels $|c\tau'|^n = \begin{vmatrix} c\tau & 0 \\ 0 & v\tau' \end{vmatrix}^n = \begin{vmatrix} (c\tau)^n & 0 \\ 0 & (v\tau')^n \end{vmatrix}$ so the particles remain non-interacting for $n > 2$, and cannot be characterized as a single valued interaction (Ref. Fermat's last theorem) as positive definite $(c\tau')^2 = (c\tau)^2 + (v\tau')^2$, but only in terms of the complex conjugate.

Non-Interacting Particles SU(2)

In this case, one particle (the "rest" particle) is real and the perturbing particle is imaginary. (In the complex plane, any two complex numbers can be represented as "connected" to the real axis equidistant and perpendicular to the "metric" between them).

$$(c\tau') = (c\tau) + i(v\tau')$$
$$(c\tau')(c\tau')^* = \left[(c\tau) + i(v\tau')\right]\left[(c\tau) - i(v\tau')\right] = (c\tau)^2 + (v\tau')^2$$

Note that this is an example of the equation of a circle for

$$r^2 = (c\tau)^2 + (v\tau')^2, \ r^2 \triangleq (c\tau')(c\tau')^*$$

This means that physical systems characterized by complex numbers (Lie Groups) factor out multiplicative interactions between particles by complex conjugation, and so eliminate the interaction energy (which makes the system non-linear in its simplest form, and thus is equivalent to eliminating Newton's law of gravity (as mass interaction).

Lie Algebra

"The Lie Algebra of SU(2) consists of 2x2 skew-Hermitian matrices with trace zero.

Explicitly this means

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\,a & -\overline{z} \\ z & -i\,a \end{pmatrix} : a \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

The Lie Algebra of SU(2) is generated by the following matrices

$u_1 = \begin{vmatrix} 0 & i \\ i & 0 \end{vmatrix}, \quad u_2 = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}, \quad u_3 = \begin{vmatrix} i & 0 \\ 0 & -i \end{vmatrix}$, which have the form of the general element specified above." – Wikipedia

$Det|u_1| = i^2 = -1 \neq -1$ since $2 = \log_i\left(i^2\right) = \log_i(-1) \neq 1 = \log_1(-1)$

$Tr|u_2| = 0, \quad Det|u_2| = 1^2$, so $|u_2|$ represents the product between two particles that do not exist.

$Tr|u_3| = 0, \; Det|u_3| = -i^2 = 1$ so $|u_3|$ represents equal and opposite imaginary numbers which do not exist $i = -i \Leftrightarrow i - i = 0 = Tr|u_3|$.

Pauli Matrices

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Tr(\sigma_1) = 0,\ Det(\sigma_1) = -1^2$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ Tr(\sigma_2) = 0,\ Det(\sigma_2) = 1$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ Tr(\sigma_3) = 0,\ Det(\sigma_3) = 1^2$$

$$Det(\sigma_1) + Det(\sigma_3) = 0^2$$

$$|\sigma_{1+2}| \triangleq \sigma_1 + \sigma_2 = \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix},\ Tr|\sigma_{1+2}| = 0, Det|\sigma_{1+2}| = -(1^2 + i^2) \neq -(1^2 + 1^2) = -2(1^2)$$

σ_1 represents a negative interaction energy (so not positive definite) of two particles that do not exist, since

$$|\sigma_1|^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1^2 & 0 \\ 0 & 1^2 \end{pmatrix}$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$(\sigma_2)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

represents a positive interaction energy between imaginary particles that do not exist.

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\sigma_3)^2 = \begin{pmatrix} 1^2 & 0 \\ 0 & (-1)^2 \end{pmatrix} = \begin{pmatrix} 1^2 & \\ & 1^2 \end{pmatrix}$$

σ_3 simply represents equality between the two particles in first order, which is also equal in second order.

$$|\sigma_{1+2}| \triangleq \sigma_1 + \sigma_2 = \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix},\ Tr|\sigma_{1+2}| = 0, Det|\sigma_{1+2}| = -(1^2 + i^2) \neq -(1^2 + 1^2) = -2(1^2)$$

$$|\psi| \triangleq \begin{pmatrix} 0 & 1-i \\ -(1+i) & 0 \end{pmatrix},\ Tr|\psi| = 0,\ Det|\psi| = (1^2 + i^2) \neq (1^2 + 1^2) = 2(1^2)$$

The Pauli matrices represents an imaginary interaction (in the Stern-Gerlach experiment, the B field is imaginary (as a particle), and is eliminated by complex conjgation, but the final particle count is the same as the initial particle count) between two particles.

The correct interpretation is derived by considering the energy of an equal and opposite Lorentz force that includes the B field explicitly). "Negative values" of energy are only possible (holes and electrons) are only possible if a larger positive mass is included (a semiconductor, the atmosphere) so that the electrons are "less positive" relative to the Fermi level in the context of the whole system.

The discussion can be referenced to coordinate representation in two dimensions by renaming

$$a \triangleq x \triangleq c\tau,\ b \triangleq x' \triangleq y = v\tau'$$

In the relation $f' \cong \frac{x}{y} = \frac{x}{x'}$, $y = x' = 0 \Leftrightarrow f' = x$, since the dimension characterized by $y \triangleq t$ does not exist.

Note that:

$$d \triangleq x + t$$
$$d^2 = (x+t)^2 = x^2 + t^2 + 2xt$$
$$d \triangleq x + \frac{1}{t}$$
$$d^2 = \left(x + \frac{1}{t}\right)^2 = x^2 + \left(\frac{1}{t}\right)^2 + 2x\frac{1}{t},\ 2x\frac{1}{t} = 2\left(\frac{x}{t}\right) \triangleq 2v,\ v = \frac{x}{t}$$

$$d \triangleq x + it$$
$$dd^* = (x+it)(x-it) = x^2 + t^2$$

$$d \triangleq x + \left(\frac{i}{t}\right)$$
$$dd^* = \left(x + \frac{i}{t}\right)\left(x - \frac{i}{t}\right) = x^2 + \left(\frac{1}{t}\right)^2$$