

The Pauli Matrices and SU(2)

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02/18/2024

Updated: 3/2/2024

[Unitary Matrix](#) (Wikipedia)

[Fermions](#)

[Working Document](#)

If existence # is defined as $1 + 0 = 1$, $1 > 0$ so that $1 - 1 = 0$, then the group is "Special" because it cannot exist. That is, it is "much ado about nothing", or "all hat but no cattle".

A complex matrix σ_i , $i = 1, 2, 3$ is Special Unitary if it is unitary and its matrix determinant = 1

Bertrand Russell's Paradox - "A barber in a village shaves all those, and only those, that don't shave themselves. Does the barber shave himself?" Solution: A barber cannot both shave himself and not shave himself (at the same time); such a barber cannot exist. Mathematically, this is equivalent to the expression $1^2 \neq 1$

Proof:

$$\#(\sqrt{1}) := \sqrt{1} + 0 = \sqrt{1}$$

$$\#(\sqrt{2}) = \sqrt{1} + \sqrt{1} = 2(\sqrt{1})$$

$$\left[\#(\sqrt{2})\right]^2 = (\sqrt{1} + \sqrt{1})^2 = \left[(\sqrt{1})^2 + (\sqrt{1})^2\right] + \left[2(\sqrt{1})(\sqrt{1})\right] = 4$$

Where $\left[(\sqrt{1})^2 + (\sqrt{1})^2\right] = [1 + 1] = 2$ represents the existence of interacting (multiplying, "entangled")

units and $\left[2(\sqrt{1})(\sqrt{1})\right] = 2(\sqrt{1})^2 = 2$ represents the interaction itself. $2(1^2) = [1^2 + 1^2] \neq 4(1^2)$. Note

that $\left[(\sqrt{1})^2 + (\sqrt{1})^2\right] = [1 + 1] = 2(1)^2 \neq 1$ and $\left[2(\sqrt{1})(\sqrt{1})\right] = 2(\sqrt{1})^2 = 2(1)^2 \neq 2(1)$.

$$2(1)^2 \neq 2(1) \leftrightarrow (1)^2 \neq (1), \text{ QED}$$

This result means that two elements must exist before they can be multiplied, where multiplication is represented by the interaction term

This relationship is represented in terms of 2×2 matrices as the existence of two interacting elements where:

$$\begin{aligned} \left[\#((\sqrt{2})^2) \right] &= 2^2 = \text{Tr} \begin{vmatrix} (\sqrt{1})^2 & 0 \\ 0 & (\sqrt{1})^2 \end{vmatrix} + \text{Det} \begin{vmatrix} (\sqrt{1}) & (\sqrt{1}) \\ -(\sqrt{1}) & (\sqrt{1})^2 \end{vmatrix} \\ &= \text{Tr} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \text{Det} \begin{vmatrix} \sqrt{1} & \sqrt{1} \\ -\sqrt{1} & \sqrt{1} \end{vmatrix} = [(1)+(1)] + 2(1) = 4(1) \end{aligned}$$

. A similar analysis can be performed for

$$\#(1) := 1 + 0 = 1$$

$$\#(2) = 1 + 1 = 2$$

$$\left[\#(2) \right]^2 = (1+1)^2 = [1^2 + 1^2] + [2(1)(1)] = 4(1^2)$$

4 Dimensions

Note that this is not equivalent to the 4D matrix:

$$\begin{bmatrix} (\sqrt{1}) & 0 & 0 & 0 \\ 0 & (\sqrt{1}) & 0 & 0 \\ 0 & 0 & (\sqrt{1}) & 0 \\ 0 & 0 & 0 & (\sqrt{1}) \end{bmatrix}^2 = \begin{bmatrix} (\sqrt{1})^2 & 0 & 0 & 0 \\ 0 & (\sqrt{1})^2 & 0 & 0 \\ 0 & 0 & (\sqrt{1})^2 & 0 \\ 0 & 0 & 0 & (\sqrt{1})^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ which}$$

characterizes the existence of 4 elements without interaction.

The Pauli Matrices

The Pauli Matrices are represented as spanning the Special Group $SU(2)$, and were proposed by Wolfgang Pauli as a characterization of the result of the Stern-Gerlach experiment, in which fermions were discovered. The Pauli Matrices consist of a set of three 2x2 matrices $\{|\sigma_1|, |\sigma_2|, |\sigma_3|\}$ defined by the relations:

$$|\sigma_1| := \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad |\sigma_2| := \begin{vmatrix} 0 & i \\ -i & 0 \end{vmatrix} = \begin{vmatrix} 0 & (\sqrt{-1}) \\ -(\sqrt{-1}) & 0 \end{vmatrix}, \quad i := \sqrt{-1}, \quad \text{and} \quad |\sigma_3| := \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}.$$

[Unitary Matrix](#) (Wikipedia)

The fact that the group operation is multiplicative but not additive means that the elements of the group do not exist.

The Identity Matrix (existence and non-interaction)

$$|I| := \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

The Identity Matrix asserts the existence and non-interaction of two elements, where

$$Tr|I| := Tr \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 + 1 = 2; \text{ two elements exist}$$

$$Det|I| := Det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1^2; \text{ a single particle interacts with itself (as an equal and opposite "force").}$$

$$|\sigma_1| := \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$Tr(|\sigma_1|) := Tr \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 + 0; \text{ its additive elements do not exist.}$$

$$Det(|\sigma_1|) := Det \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -(1^2) < 0; \text{ its interactive elements are negative and do not exist.}$$

$$Tr(|\sigma_2\rangle) = Tr \begin{vmatrix} 0 & (\sqrt{-1}) \\ -(\sqrt{-1}) & 0 \end{vmatrix} = Tr \begin{vmatrix} 0 & i \\ -i & 0 \end{vmatrix} = 0 + 0 = 0; \text{its additive elements do not exist.}$$

$$Det(|\sigma_2\rangle) = Det \begin{vmatrix} 0 & (\sqrt{-1}) \\ -(\sqrt{-1}) & 0 \end{vmatrix} = -[(\sqrt{-1})][-(\sqrt{-1})] = -[-(-\sqrt{-1})\sqrt{-1}];$$

$$= (\sqrt{-1})(\sqrt{-1}) = (\sqrt{-1})^2 = i^2$$

; only an imaginary interaction exists.

$$|\sigma_3\rangle := \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

The interpretation of $|\sigma_3\rangle$ as \pm charge or mass is misleading, since it only asserts that IF there are two elements, they are equal.

$$Tr(|\sigma_3\rangle) := Tr \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = 1 - 1 = 0 ; \text{its additive elements are equal but existence is irrelevant.}$$

$$Det(|\sigma_3\rangle) := Det \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -(1^2) < 0 ; \text{the product of additive elements is negative, and therefore do not exist.}$$

Note that additively, if the order of the vectors of $|\sigma_3\rangle$ are changed so that

$$|\sigma_3|_{(\leftrightarrow)} := \left| \frac{1}{-1} \right| + \left| \frac{1}{1} \right| = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \text{ then } Tr(|\sigma_3|_{(\leftrightarrow)})^2 = Tr \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}^2 = [1^2 + 1^2], \text{ } Det|\sigma_3|_{(\leftrightarrow)} = [2(1)(1)]$$

So that

$$\# = (1+1)$$

$$\#^2 = (1+1)^2 = \left[\left(Tr|\sigma_3|_{(\leftrightarrow)} \right)^2 \right] + \left[Det|\sigma_3|_{(\leftrightarrow)} \right] = [1^2 + 1^2] + [2(1)(1)]$$

(Physically, this represents equal charges or equal masses or velocities where an invariant force f

represents a momentum (as a prime number) so

$$f_v := mv = mv \left(\frac{mv}{mv} \right) := (mv)1_{(mv)} > 0$$

$$f_v \rightarrow m_0 \leftarrow f_v, (f_v^2)_{(x=0)} = m_0$$

is interpreted to mean that equal and opposite force elements in a vacuum where there is no acceleration/interaction) are approaching each other, where their equal and opposite interaction characterizes a rest mass.

$$|\sigma_3|^2 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}^2 = \begin{vmatrix} 1^2 & 0 \\ 0 & 1^2 \end{vmatrix} \text{ where } Det \begin{vmatrix} 1^2 & 0 \\ 0 & 1^2 \end{vmatrix} = (1^2 + 1^2) = 2(1^2) \neq Det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1^2 \text{ so } |\sigma_3| \notin \left\{ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right\}$$

where it should again be emphasized that $2(1^2) = (1^2 + 1^2)$ represents existence, not interaction.

Complex Conjugation

Complex Conjugation introduces real and imaginary existence into imaginary numbers, where

$$\psi := \sqrt{1} + i$$

$$\psi^* := \sqrt{1} - i$$

$$\psi\psi^* = (\sqrt{1} + i)(\sqrt{1} - i) = [1 + 1] + [i\sqrt{1} - (\sqrt{1})i]$$

Where $\sqrt{1}$ is specified instead of 1 as the real term. This represents the existence of Schroedinger and his imaginary cat when not directly observed (i.e., in the box, dead or not)

Sum of Pauli Matrices

$$|\Sigma| := |\sigma_1 + \sigma_2 + \sqrt{\sigma_3}| = \begin{vmatrix} (\sqrt{1}) & (\sqrt{1}+i) \\ (\sqrt{1}-i) & (\sqrt{-1}) \end{vmatrix} = \begin{vmatrix} \sqrt{1} & \psi \\ \psi^* & \sqrt{-1} \end{vmatrix}, \psi := \sqrt{1} + i$$

$$|\Sigma|^2 = \begin{vmatrix} (\sqrt{1}) & \psi \\ \psi^* & i \end{vmatrix} \begin{vmatrix} (\sqrt{1}) & \psi \\ \psi^* & i \end{vmatrix} = \begin{vmatrix} (1+\psi\psi^*) & ((\sqrt{1})(\psi)+\psi i) \\ ((\sqrt{1})\psi^*+i\psi^*) & (i^2+\psi\psi^*) \end{vmatrix}$$

$$\text{Tr}|\Sigma|^2 = 2(\psi\psi^*)$$

$$\begin{aligned} \text{Det}|\Sigma|^2 &= (1+\psi\psi^*)(i^2+\psi\psi^*) - [((\sqrt{1})(\psi)+i\psi)][((\sqrt{1})\psi^*+i\psi^*)] \\ &= [(1)(i^2)+(\psi\psi^*)^2] + [(\psi\psi^*)(i^2)+(\psi\psi^*)(1)] - \\ &[(1)(\psi\psi^*)+(\psi i)(i\psi^*)] + [(i\psi)(\sqrt{1}\psi^*)+(i\psi^*)(\sqrt{1}\psi)] \\ &= [(1)(\psi\psi^*)+(i^2)(\psi)(\psi^*)] + [(i\psi)(\sqrt{1}\psi^*)+(i\psi^*)(\sqrt{1}\psi)] \\ &= [(\psi\psi^*)-(\psi\psi^*)] + (i\sqrt{1})[(\psi)(\psi^*)+(\psi^*)(\psi)] \\ &= [0] + (i\sqrt{1})[(1+1)+(1+1)] = [0] + 2((\sqrt{-1})\sqrt{1})(1+1) \\ &= 2(1)(2) = 4 \sin\left(\frac{\pi}{4}\right) \end{aligned}$$

$$\# := (\sqrt{1} + \sqrt{1})$$

$$\#^2 = (\sqrt{1} + \sqrt{1})^2 = [(\sqrt{1})^2 + (\sqrt{1})^2] + [2(\sqrt{1})(\sqrt{1})] = [1+1] + [2(1)] = 4 \cos(0)$$