Groups and Vectors

Charles Keyser

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Working Document

An element is represented by a positive real number a. Every number is prime relative to its own base $a := a\left(\frac{a}{a}\right) = a\left(1_a\right)$ where $\left(\frac{a}{a}\right) = \tan\left(45^\circ\right) = \tan\left(\frac{\pi}{8}\right)$ within the unit circle.

Every even number is the sum of two primes $n + n = 2n$; therefore n must be odd.

An element of reality is represented in one dimension by a positive number $a = \int_0^a da$ where $a + 0 = a$, where the addition operation "$+$" represents existence.

Groups

A group consists of two identical elements and the operation of multiplication "$\times$, juxtaposition". The definition of a group is consistent under self-multiplication (exponentiation) in first order since $a = \left(\sqrt[n]{a}\right)^n$ for all n, and in particular $a = \sqrt{a \times a} = \left(\sqrt{a}\right)\left(\sqrt{a}\right) = \sqrt{a^2} = \left(\sqrt{a}\right)^2$ Note that the operation of addition $+$ (existence) is excluded from the definition of a group; i.e., the expression $a + a = 2a$ is not a group, so that a group is represented by odd numbers only.

The group a is defined as first order.

The transition of a group from $a = 0$ to $a + 0 = a$ (nothing to existence) can be represented by the group relation $f(a) = a\sin\theta = \left(a\sin\theta\right)\frac{\left(a\sin\theta\right)}{\left(a\sin\theta\right)} = \left(a\sin\theta\right)1_{\left(a\sin\theta\right)}$, $\sin\theta = \sin\theta\frac{\sin\theta}{\sin\theta} = 1_{\sin\theta}$, where $0 \le \theta \le \frac{\pi}{4}$ so that a ranges from 0 to a, and the null group is defined by $0 = 0\left(\frac{0}{0}\right) = 0\left(1_0\right)$

The expression $a^2 = a^2 + 0^2$ is defined as second order; which is inconsistent with the concept of group, since its definition requires the existence of a second group, where

$$\begin{aligned} \# &= (a+a) = 2a \\ \#^2 &= (a+a)^2 = \left[a^2 + a^2\right] + \left[2a^2\right]. \end{aligned}$$

The term $\#^2$ includes an existence term $\left[a^2 + a^2\right]$ and an interaction term $\left[2a^2\right]$ where the multiplicative term only appears in the "interaction, entanglement," term $\left[2a^2\right]$, so the term a^2 does not represent a group. However, the "count" term $\#$ appears as a group in both first and second order, where $\# = \#\left(\frac{\#}{\#}\right)$ and $\#^2 = \#^2\left(\frac{\#^2}{\#^2}\right)$.

Transition Group ($\sin\theta$)**for two interacting elements**

$$\sin\theta = \sin\theta\left(\frac{\sin\theta}{\sin\theta}\right) = \sin\theta\left(1_{\sin\theta}\right)$$

$$1 = \cos(0) = \cos\left(\pi\right)$$

$$\varphi := \cos(0) + \sin\theta$$

$$\varphi^2 = \left(\cos(0) + \sin\theta\right)^2 = \left[\cos^2(0) + \sin^2\theta\right] + \left[2(\cos(0)(\sin\theta)\right] = \left[1^2 + \sin^2\theta\right] + \left[2\left(1\right)(\sin\theta)\right]$$

, where $\left[1^2 + \sin^2\theta\right]$ is the existence term and $\left[2(1)(\sin\theta\right]$ is the interaction term.

For $\theta = 0$, $\varphi := 1$, $\varphi^2 = 1^2$, for $\theta = \pm\frac{\pi}{4}$, $\varphi := 1 + 1$, $\varphi^2 := \left(1+1\right)^2 = 4(1^2)$

However, for

$$\psi := \left[1 + i\sin\theta\right]$$

$$\psi^* := \left[1 - i\sin\theta\right]$$

$$\psi\psi^* := \left[1 + i\sin\theta\right]\left[1 - i\sin\theta\right] = \left[1^2 + \sin^2\theta\right] + \left(1\right)\left(i\sin\theta\right) - \left(1\right)\left(i\sin\theta\right)$$

$$\psi\psi^* = \left[1^2 + \sin^2\theta\right]$$

For $\theta = 0$, $\psi = \psi^* = 1$, $\psi\psi^* = 1^2$, for $\theta = \pm\frac{\pi}{4}$, $\psi\psi^* := 1^2 + 1^2 \neq \varphi^2 = 4(1^2)$

Note that if the second term is not identical to the first, the expressions are still valid (if a distinction between groups is to be emphasized, a different color for each will be used):

$$\# = (a + b)$$

$$\#^2 = (a + b)^2 = \left[a^2 + b^2\right] + \left[2ab\right]$$

(An expression of Fermat's Last Theorem $\forall\left\{a, b, c, n : n > 1\right\} c^n \neq a^n + b^n$ for the case $n \geq 2$ via the Binomial Expansion.)

$$\# = (a+b)$$
$$\#^2 = (a+b)^2 = \left[a^2 + b^2\right] + \left[2ab\right]$$

Parametrization

Groups can be parametrized: e.g. $a := ct,\ b := vt'$

In physics, first order groups represent forces, second order groups represent physical quantities expressed as forces. For example, a rest mass (or charge) can be represented as the interaction of two equal and opposite forces, e.g.

$$f_{(m_0)} := \frac{f_{(m_0)}}{2} + \frac{f_{(m_0)}}{2}$$
$$m_0 = \left(f_{(m_0)}\right)^2 = \left(\frac{f_{(m_0)}}{2} + \frac{f_{(m_0)}}{2}\right)^2 = \left[\left(\frac{f_{(m_0)}}{2}\right)^2 + \left(\frac{f_{(m_0)}}{2}\right)^2\right] + \left[2\left(\frac{f_{(m_0)}}{2}\right)\left(\frac{f_{(m_0)}}{2}\right)\right]$$

1. Force due to Length $x = vt$ (“field”)
2. Force $f = ct$ (relativistic force)
3. Electromagnetism $f_c = EB = F_{\mu\nu}F^{\mu\nu}$
4. Complementary variables Px, Et

Pythagorean Triples

In the following analysis, the Pythagorean Triple $\{c,a,b\} = \{5,4,3\}$ will be used; the results will be valid for all Pythagorean Triples. Note that $5 = \left(\sqrt{5}\right)\left(\sqrt{5}\right),\ 4 = \left(\sqrt{4}\right)\left(\sqrt{4}\right), 3 = \left(\sqrt{3}\right)\left(\sqrt{3}\right)$ so that each element represents a group, taken individually.

Consider the expression $\# := 7 = 4 + 3$, where $7 = \left(\sqrt{7}\right)\left(\sqrt{7}\right)$ but that 4 and 3 no longer represent groups within the sum, since $\#^2 = 7^2 = 49 \neq 4^2 + 3^2 = 25 = c^2 = 5^2$.

Expanding this expression results in:

$\#^2 = 49 = \left[4^2 + 3^2\right] + \left[2(4)(3)\right] = [16+9] + 2[12]$ so that

$$\#^2 = 49 = [25] + [24]$$

Note that the equation of a circle $5^2 = 4^2 + 3^2$ cannot be derived from $7 = 4 + 3$ In order to derive this expression imaginary (complex) numbers must be used (complex numbers represented by the color magenta):

$i := \sqrt{-1}$
$\psi := [4 + 3i]$
$\psi^* := [4 - 3i]$
$\psi\psi^* = [4+3i][4-3i] = \left[4^2 - (3i)^2\right] + 4(3i) - 4(3i)$

So that

$$\psi\psi^* = [4+3][4-3] = \left[4^2 + (3)^2\right] = [16+9] = 25$$

That is, $\psi\psi^* = 25$

Inserting this value into the real expression results in

$$\#^2 = 49 = [\psi\psi^*] + \left[2(4)(3)\right] = [25] + [24],$$

so that the expressions are numerically equivalent in the group representation $\#^2 = \#^2$ but that $\psi\psi^* \neq \#^2 \left(= \#^2\right)$

The results obtained above have profound consequences in the expression of geometrical physics (Classical Electromagnetism, Special and General Theory of Relativity, Quantum Mechanics, Quantum Field Theory, which will be explored below) In order to understand this, the concept of groups must be explored in relation to the results obtained above.

The representation of the existence (addition, $+$) of two group elements $\{4,3\}$ is characterized by the first order relation $\{\#, a, b\} \equiv \{7,4,3\}$ where $\# := 7 = 4 + 3 \leftrightarrow \left(\sqrt{7}\right)^2 = \left(\sqrt{4}\right)^2 + \left(\sqrt{3}\right)^2$, defined by the groups $\left(\sqrt{7}\right)\times\left(\sqrt{7}\right) = 7$, $\left(\sqrt{4}\right)\times\left(\sqrt{4}\right) = 4$ and $\left(\sqrt{3}\right)\times\left(\sqrt{3}\right) = 3$.

Existence

In matrix notation, this first order existence representation has the form

$$
\begin{aligned}
&Tr\left|7\right| = Tr\left|4\right| + Tr\left|3\right| \\
&= Det\left|7\right| = Det\left|4\right| + Det\left|3\right| \\
&= Det\left|4+3\right|
\end{aligned}
$$

Interaction (Multiplication, “entropy”, “entanglement”)

If the groups are characterized as vectors $Tr\begin{vmatrix}4\\0\end{vmatrix}=4$ and $Tr\left|\frac{0}{3}\right|=3$, the two vectors can be represented by the 2×2 matrix $\begin{vmatrix}4 & 0\\0 & 3\end{vmatrix}$ where $Tr\begin{vmatrix}4 & 0\\0 & 3\end{vmatrix}=7, Det\begin{vmatrix}4 & 0\\0 & 3\end{vmatrix}=12$, but $\left|\frac{4}{0}\right|\cdot\left|\frac{0}{3}\right|=\vec{4}\cdot\overleftarrow{3}=0$ (component by component multiplication, the vector “dot” product,so that the vector $\left|\frac{0}{3}\right|$ must be imaginary.)

Then $Tr\begin{vmatrix}4 & 0\\0 & 3\end{vmatrix}=\vec{4}\cdot\overleftarrow{3}=\overleftrightarrow{12}$ and $Tr\begin{vmatrix}4 & 0\\0 & 3\end{vmatrix}^2=Tr\begin{vmatrix}16 & 0\\0 & 9\end{vmatrix}^2=25$ represents the concept of orthogonality between the imaginary and real axes, so that $4\perp 3$; the vectors are affine (do not interact), and the product is not defined. This corresponds to multiplication without existence.

However, in second order, if multiplication (interaction) is defined, then the relation

$\#^2=49=\left[4^2+3^2\right]+\left[2(4)(3)\right]=[16+9]+2[12]$ is represented by the matrix relations

$$\#^2=(\#)(\#)=(7)(7)=Tr\begin{vmatrix}4^2 & 0\\0 & 3^2\end{vmatrix}+Det\begin{vmatrix}4 & 4\\-3 & 3\end{vmatrix}=\left[4^2+3^2\right]+\left[2(4)(3)\right]=[25]+[24]=49$$

Where the term $Tr\begin{vmatrix}4^2 & 0\\0 & 3^2\end{vmatrix}$ represents the existence of the interacting elements and the term $Det\begin{vmatrix}4 & 4\\-3 & 3\end{vmatrix}$ represents their interaction. Note that these two terms are not groups, since addition relates them, but $\#$ is a group, since $\#^2=(\#)(\#)$

The analysis in this section shows that in order for multiplication between groups to occur, their existence must first be established; for the case $a=b=1$, $\#=1+1=2$ but
$\#^2=\left[1^2+1^2\right]+\left[2(1^2)\right]=4(1^2)$.

The above representation is not equivalent to

$$Tr\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 4(1) = 1+1+1+1$$

$$Tr\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}^2 = Tr\begin{vmatrix} 1^2 & 0 & 0 & 0 \\ 0 & 1^2 & 0 & 0 \\ 0 & 0 & 1^2 & 0 \\ 0 & 0 & 0 & 1^2 \end{vmatrix} = 4(1^2) = 1^2+1^2+1^2+1^2$$

, which represents existence without interaction (no multiplication between colors.

Note that the Minkowski metric is represented by

$$|M| := \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & i^2 & 0 & 0 \\ 0 & 0 & i^2 & 0 \\ 0 & 0 & 0 & i^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & i^2 & 0 & 0 \\ 0 & 0 & i^2 & 0 \\ 0 & 0 & 0 & i^2 \end{vmatrix}$$

This distinction imply profound consequences in the modern interpretation of mathematical physics.

The Identity Matrix and the Special Unitary Group SU(2)

The most important matrix representation of the foundation of mathematical physics consists of the Identity Matrix and the Special Unitary Group SU(2). A group consists of two group elements and an operator. The operators defined for groups are the existence operator addition $+$, multiplication $\times$ (or juxtaposition) The sum of two elements is represented by the $\#$ symbol

A single unit group element

The existence of a single unit group element in first order $\# := 1+0=1$ is represented by the matrix $|1|,\ Tr|1|=1,\ Det|1|=1$. Its second order representation is $|1|^2 = 1^2$ Note existence of the element represented by the zero element (0) which represents non-existence.

Two group elements with the existence operator $+$

First order representation

The existence of two unit group elements with the addition operation is represented by the relation:

$\# = 1+1$, which has the matrix representation:

$|I| := \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}$, $Tr|I| = 1+1,\ Det|I| = 1^2$ The Identity Matrix can be represented as the component by component sum of two vectors: $\begin{vmatrix}1\\ 0\end{vmatrix} + \begin{vmatrix}0\\ 1\end{vmatrix} = \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}$ where the order of addition is uniquely defined.

Second order representation

The second order representation the sum is

$\#^2 = (1+1)^2 = \left[1^2+1^2\right]+\left[2(1)(1)\right]$ where the sum now includes both and existence term $\left[1^2+1^2\right]$ and a multiplication term $\left[2(1)(1)\right]$ where both must exist. Note that the second order sum $\#^2 = (\#)(\#)$ is a group, but the sum of the individual terms in the representation are not, since they involve two operations, $+$ and $\times$. The second order matrix representation is given by:

$$\#^2 = Tr\begin{vmatrix}1^2 & 0\\ 0 & 1^2\end{vmatrix} + Det\begin{vmatrix}1 & 1\\ -1 & 1\end{vmatrix} = \left[1^2+1^2\right]+\left[2\left(1^2\right)\left(1^2\right)\right] = 4(1^2)$$

Note that the interaction matrix $\begin{vmatrix}1 & 1\\ -1 & 1\end{vmatrix} = \begin{vmatrix}1\\ -1\end{vmatrix} + \begin{vmatrix}1\\ 1\end{vmatrix} = \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} + \begin{vmatrix}0 & 1\\ -1 & 0\end{vmatrix}$

The Special Unitary Group SU(2)

The Special Unitary Group consists of the three Pauli matrices, which were first proposed as a representation of the results of the Stern-Gerlach experiment, which produced fermions.

These matrices are given by

$$|\sigma_1| := \begin{vmatrix}0 & 1\\1 & 0\end{vmatrix} = \begin{vmatrix}0\\1\end{vmatrix} + \begin{vmatrix}1\\0\end{vmatrix},\ Tr|\sigma_1| = 0,\ \ Det|\sigma_1| = -1^2$$

$$|\sigma_2| := \begin{vmatrix}0 & i\\-i & 0\end{vmatrix} = \begin{vmatrix}0\\-i\end{vmatrix} + \begin{vmatrix}i\\0\end{vmatrix} = \begin{vmatrix}0\\-i\end{vmatrix} + \begin{vmatrix}i\\0\end{vmatrix}\ Tr|\sigma_2| = 0,\ Det|\sigma_2| = -i^2 = \left(i^2\right)\left(i^2\right) = (-1)(-1) = i^4 = 1$$

$$|\sigma_2| := \begin{vmatrix}0 & i\\-i & 0\end{vmatrix},\ Tr|\sigma_2| = 0,\ Det|\sigma_2| = -i^2 = i^4 = 1$$

The Equality Matrix $|\sigma_3|$

$|\sigma_3| := \begin{vmatrix}1 & 0\\0 & -1\end{vmatrix} = \begin{vmatrix}1\\0\end{vmatrix} + \begin{vmatrix}0\\-1\end{vmatrix}, Tr|\sigma_3| = 1 - 1 = 0,\ \ Det|\sigma_3| = -1^2$ Note that $Tr|\sigma_3| = 0$ represents equality in first order, while $Det|I| + Det|\sigma_3| = 1^2 - 1^2 = 0$ represents equality in second order.

Complex Conjugation

Note that if $\psi := i,\ \psi^* := -i$ then

$|\sigma_2| := \begin{vmatrix}0 & i\\-i & 0\end{vmatrix} := \begin{vmatrix}0 & \psi\\-\psi^* & 0\end{vmatrix}$ so that $Det|\sigma_2| = \psi\psi^* = 1$

However, note that if

$$\psi := (1 + i),\ \psi^* := (1 - i)$$
$$\psi\psi^* := (1 + i)(1 - i) = \left[1^2 + 1\right] + (1i) - (1i)$$

, so that $\psi\psi^* = \left[1^2 + 1^2\right]$ where the multiplication terms $+(1i) - (1i)$ have been eliminated by conjugation, then $|\#| = |\sigma_1| + |\sigma_2| = \begin{vmatrix}0 & 1+i\\1-i & 0\end{vmatrix} := \begin{vmatrix}0 & \psi\\\psi^* & 0\end{vmatrix}, Det|\#| = \psi\psi^* = 1^2 + 1$

Note that the real term and the imaginary term are of different order. This can be corrected by re-defining

$|\sigma_2| := \begin{vmatrix} 0 & -i^2 \\ i^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$, $Det|\sigma_2| = 1^2$ and

$|\sigma_2|^2 := \begin{vmatrix} 0 & i^4 \\ -i^4 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$ so that $|\#| = |\sigma_1| + |\sigma_2| = \begin{vmatrix} 0 & 1+1 \\ 1-1 & 0 \end{vmatrix} := \begin{vmatrix} 0 & \psi \\ \psi^* & 0 \end{vmatrix}$, $Det|\#| = \psi\psi^* = 1^2 + 1^2$.

(This correction is the result of Russell's Paradox – "A barber in a village shaves all those and only those that don't shave themselves. Does the barber shave himself?" – Bertrand Russell. Ans: there is no such barber; a barber cannot both shave and not shave himself. Mathematically, $1^2 \neq 1 \leftrightarrow 1 = \sqrt{1^2} \neq \sqrt{1}$.

Then
$\#^2 = \left[1^2 + 1^2\right] + 2(1)(1)$ is equivalent to $\#^2 = \left[1^2 + 1^2\right] + 2(1)(1) = [\psi\psi^*] + 2(1)(1)$ and the count is preserved.

Vector Products in two dimensions

Vector products in two dimensions are represented by 2x2 matrices, where the identity matrix characterizes the existence of two vectors under the group operation of addition.

$|I| := \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix} + \begin{vmatrix} 0 \\ 1 \end{vmatrix}$ where $Tr|I| = 1 + 1$ and $Det|I| = 1^2$ where the Trace means that both elements exist and the Determinant means that the product of the elements exist in second order.

The "dot" product of two vectors consists of a component-by-component product of the elements of the vectors, so that

$\begin{vmatrix} 1 \\ 0 \end{vmatrix} \cdot \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$, $\begin{vmatrix} 1 \\ 0 \end{vmatrix} \cdot \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 1^2 \\ 0 \end{vmatrix}$, and $\begin{vmatrix} 0 \\ 1 \end{vmatrix} \cdot \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ 1^2 \end{vmatrix}$

The cross product of two vectors consists of a mapping of a two dimensional vector into one dimension of second order:

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1^2 \end{vmatrix} \equiv 1 \otimes 1 = 1^2$ where $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1^2 \end{vmatrix} \equiv 1 \otimes 1 = -1^2$

Note that the cross product eliminates the original vectors, so that $\overrightarrow{1} \otimes \overleftarrow{1} = \widehat{1^2} = \overrightarrow{1} \otimes \overleftarrow{1} + \overrightarrow{1} \cdot \overleftarrow{1}$ where $\overrightarrow{1} \cdot \overleftarrow{1} = \overleftarrow{1} \cdot \overrightarrow{1} = \overleftrightarrow{0}$

For $i := \sqrt{-1}$, $i^2 = -1$, the second cross product with the 2D vectors interchanged maps two real elements on to the imaginary axis:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i^2 1^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1^2 \end{vmatrix} \equiv 1 \otimes 1 = -1^2$$, and $-1^2 = (i1)^2$

Prime Numbers

$\begin{vmatrix}1\\0\end{vmatrix}\cdot\left|\frac{1/1}{0}\right|=\frac{(1)(1_1)}{0}$ so that the element product $(1)(1_1)=1\left(\frac{1}{1}\right)=1$ defines a prime number in one dimension, and similarly for $n\begin{vmatrix}1\\0\end{vmatrix}\cdot\left|\frac{1/1}{0}\right|=\left|\frac{n(1)(1_1)}{0}\right|$ where $n=n\left(\frac{n}{n}\right)$ so that

$n(1)=n(1)\left(\frac{n(1)}{n(1)}\right)=n(1)1_{n(1)}$ is also prime and in particular for $n=1$, $1^2=1^2\left(\frac{1^2}{1^2}\right)=1^2\left(1_{1^2}\right)$ is also prime.

Note that if $n=ct$, then the product ct is also prime, but for $x=vt\rightarrow\frac{x}{t}=v\frac{t}{t}$, $v\neq t$, v and t are not prime.

However, for the interaction equation where # is interpreted as the count of elements:

$$\begin{aligned}&\#:=ct'=ct+vt'\\&\#^2=(ct')^2=(ct+vt')^2=\left[(ct)^2+(vt')^2\right]+\left[2(ct)(vt')\right]\end{aligned}$$

The count is preserved under both addition (“existence”) and multiplication (Interaction, Entanglement, Entropy, etc.) Note that $\#^2=(\#)(\#)$ is a group under mutliplication, but that $(ct')^2$ is not, since it includes both operations.