The Barber Paradox

(and Mathematical Physics)

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Russell's Paradox (Wikipedia)

Let $R=\{x\,||\,x\notin x\}$, then $R\in R\Leftrightarrow R\notin R$.

That is, $x\notin x\Leftrightarrow\{x\}\equiv\{0\}$ (my interpretation)

The Barber Paradox (as first-order logic) (Wikipedia)

$$(\exists x)\{person(x)\wedge(\forall y)(person(y)\Rightarrow[shaves(x,y)\Leftrightarrow -shaves(y,y)]\}$$

Alternatively (as a tautology)

$$(\exists x)[(person(x)\wedge\perp]$$
$$(\exists x)(\perp)$$
$$\perp$$

Proof of Fermat's Theorem

Proof of Goldbach's conjecture

The Relativistic Unit Circle (RUC)

Math Notes (most recent)

Relativity Page

"A barber in a village shaves all those, and only those that don't shave themselves. Does the barber shave himself?" – Barber Paradox

In the discussion that follows, I have used the variables (c,τ,v,τ',h) to show the discussion's relevance to Theory of Relativity and Quantum Mechanics. See the links above for further discussions of these topics.

(Note: hidden assumption: a single barber can only give one shave at a given time; i.e., he can't give two shaves simultaneously. That is, a single barber cannot shave himself and not be shaved at the same time.) (This restriction is at the heart of the Pauli Exclusion Principle.)

In the context that follows, addition refers to existence of an element or set where $x+0=x\ \forall x>0$ and $\{x+0\}=\{x\}\ \forall x>0$, and the difference $\Delta(x,y)=x-y$ is only defined for

$x \geq y$ and $x = y \Leftrightarrow x - y = 0$

The Barber event

A single Barber cannot be both “shaved” and “not shaved” at any given time t, and a single Barber can only be shaved once at any given time t.

Let $\tau = \left(\sqrt{\tau}\right)^2$ represent the number of Barbers that exist at any given time t

Let $c = \left(\sqrt{c}\right)^2 = \left(\sqrt{1}\right)^2 = 1$ represent the number of shaves given to himself by the barber at time t. Then the number n_b of shaving events $c\tau$ that occur by a single existing barber shaving himself at a given time t is given by $n_b = \left(\sqrt{c\tau}\right)^2 = \left(\sqrt{1^2}\right)^2 = 1$, where $c = \tau = 1$.

The Village event

A single villager cannot be both “shaved” and “not shaved” at any given time t, and a single villager can only be shaved once at any given time t.

Let $v = \sqrt{v^2}$ represent the number of villagers that exist and are not shaved given time t.

Let $\tau' = \sqrt{(\tau')^2}$ represent number of times they are not shaved at any given time t.

Then the number n_v “not shaved” events experienced by a single existing villager at a given time t is represented by the relation $n_v = \left(\sqrt{v\tau'}\right)^2 = \left(\sqrt{1^2}\right)^2 = 1$, where $v = \tau' = 1$

The Barber and Village events combined

Since the Barber shaves himself and the Villager is not shaved, the events are independent, and the event count for both events is given by $\#_{bv} = n_b + n_v = 1 + 1 = 2$

The event of the Barber “not shaving” a Villager a does not exist at a given time t, so its count is equal to zero. If the Barber shaves a Villager who doesn’t shave himself at the given time t, then the Barber doesn’t shave himself, so the count of that event is also zero.

Conclusion

This means that the barber cannot be both in the village and not in the village at the same time; that is, he can't be in two places at once. That is, the creation events of (positive definite $x = |x| = \left(\sqrt{x}\right)^2$ must be independent if they are created at the same time $x = y \Leftrightarrow y = x$. If x and y are interpreted as lengths, then if x and y are independent, they must exist at two different places at the same time, which requires two origins, such that $\Delta_{x,y} = y - x,\ y > x$. In this case, $y = x$ means that y does not exist, so that $x = x$ which then only can exist at the single origin (creation point) $x = 0$ where the existence of x is given by $x = m_0 = c\tau$ and if there is a perturbing state $y = v\tau'$, if they do not interact with each other over the distance, the count $n_{x,y} = x + y$ which is equivalent to the length $\Delta_{x,y} = x + y = n_{x,y}$, where $\Delta_{x,y}$ can now be interpreted as a "ruler" where the coordinate speed of measurement is infinite, and the mass of measurement is 0.

Therefore, the existence of $x = c\tau$ is interpreted as an initial statement (with no interaction) and is therefore represented by a prime number. The existence of $y = v\tau'$ (with no interaction) is also represented by a prime number, and as such, cannot be in the same place at the same time $c \neq v,\ \tau \neq \tau'$ $\Delta_{c\tau,v\tau'} = (c\tau) + (v\tau') = n_{c\tau,v\tau'}$ where $(c\tau) = \sqrt{(c\tau)^2}$ and $(v\tau') = \sqrt{(v\tau')^2}$ This definition includes all possible sets of positive integers defined by $n \triangleq (c\tau)$ and $n_{c\tau} \triangleq \sum_{1}^{n} (c\tau),\ (c\tau) = 1$, and similarly for $m \triangleq (v\tau')$

Interactions (shaves) between the barber(s) and the villager(s) over a period Δ t.

This interaction would be defined by the events of an existing barbers giving shaves to themselves and to an existing villagers at different times, so that $\varphi \triangleq (c\tau') = (c\tau) + (v\tau')$ at different times, if these events exist $\varphi \triangleq (c\tau') = \sqrt{(c\tau')^2} = \sqrt{\left[(c\tau) + (v\tau')\right]^2} = \sqrt{\left[(c\tau)^2 + (v\tau')^2 + 2(c\tau)(v\tau')\right]}$.

Then $\varphi^2 = (c\tau')^2 = \left[(c\tau)^2 + (v\tau')^2\right] + 2(c\tau)(v\tau')$, where $(c\tau') = (c\tau) \Leftrightarrow (v\tau') = 0$ (i.e., the village event does not exist).

The expression $(c\tau')^2 = \left[(c\tau)^2 + (v\tau')^2\right] + 2(c\tau)(v\tau')$ characterizes the changing state where the barber(s) are shaving the villagers over a period of time Δt, where the initial state and the final state implies that the interaction is characterized by $h^2 \triangleq 2(c\tau)(v\tau') = 0 \Leftrightarrow (v\tau') = 0$ so that $(c\tau')^2 = (c\tau)^2$

During the period of change, note that h^2 is changing because $(c\tau)$ is scaled by $(v\tau')$, and so therefore is not invariant.

Important: Note that $(c\tau'),(c\tau)$ and $(v\tau')$ remain positive integers; as an example,

$$7 = 3 + 4$$
$$7^2 = \left[3^2 + 4^2\right] + 2(3)(4) = [25] + 24 = 49$$

Positive and Negative Integers

Note that a group is defined as relation between two elements a and b having only one operation and thus only one result c, where the expression $c = a \odot b$ and $\odot$ represents the group operation.

The expression $c = a + b,\ \odot \equiv +$ represents (the group operation of) addition and $c = a - b,\ \odot \equiv -$ represents (the group operation of) subtraction for positive definite integers, where for any integer $n = \sqrt{n^2}$.

Consider the expression $c = a - b,\ a \geq b,\ a \geq c$, where a, b and c are positive definite. The term c is the difference between b and a; if $b = a$ then $c = 0$. Another term for c is “deficit”, and in the expression above, it is always positive if not zero.

In order distinguish it in terms of group operations, it is assigned the color of red for addition and blue for subtraction, so that $c = a + b$, $c = a - b$ and $b = a \Leftrightarrow c = c = 0$

Note that $c + c = 2a$ and $c - c = 2b$ for if both the operations are applied, so in this case the elements a and b do not form a group; i.e., the group operation $\odot$ for $(a+b) \odot \left(a - b\right) \equiv c \odot c$, is not defined for positive definite integers; in this case it is assigned the color magenta to indicate that it is “imaginary”. It is an expression of the concept that the element c cannot be both positive and negative at the same time (compare with the results of the Barber paradox above, where a represents barbers, b represents villagers, and $\odot$ represents “shaving” and “not shaving”, respectively).

Imaginary numbers

$$2=1+1$$
$$-1=1-2$$
$$\left(-1\right)^2=\left(1-2\right)^2=1^2-2(1)-(1)2+2^2=1^2-\left[2(1)+(1)2\right]+2^2$$
$$=1^2-\left[2+2\right]+\left[2\times 2\right]=1^2-\left(4_+\right)+\left(4_\times\right)=1^2\neq i^2=-1$$

That is, $\left(-1\right)^2=1^2$ (an integer cannot be both positive and negative at the same time.

This can only be true if the operations are equivalent, so that $\left(4_+\right)\equiv\left(4_\times\right)$ and cancel under addition ($+$, "creation"), so all that's left is the group operation of self-multiplication ($\times$, "power, self-interaction", "existence")

Consider the expressions

$$-1=4-5$$
$$\left(-1\right)^2=\left(4-5\right)^2=4^2-(20)-(20)+5^2=16-\left[2(20)\right]+25$$
$$=16-40+25=41-40=1$$

Then $-1=1$, so $\left(\sqrt{-1}\right)^2=\left(\sqrt{1}\right)^2=i=1$, which is the quintessential odd number.

This relation is true for any set of integers where $-1=m-n,\ n=m+1$, so that $-1=+1$, $1-1=0$ and $\sqrt{\left(-1\right)^2}=\sqrt{\left(1\right)^2}=1$. (Again, an integer cannot be positive and negative at the same time)

Note that for:

$$\psi=(a+b)$$
$$\psi\psi^*=(a+b)(a-b)=a^2-b^2$$

Where the (real) product (ab) has been eliminated by addition and subtraction; "real conjugation".

However, for $1=a+b$, a and b cannot be integers (i.e., 1, an initial or final state, $b=0$).

Matrix representation.

The operations of addition and subtraction are therefore independent, and can be represented a a matrix

$|M_{cc}| \triangleq \begin{vmatrix} c & 0 \\ 0 & c \end{vmatrix} = \begin{vmatrix} a+b & 0 \\ 0 & a-b \end{vmatrix}$, consisting of the vectors $\left|\frac{c}{0}\right| = \left|\frac{a+b}{0}\right|$ and $\left|\frac{0}{c}\right| = \left|\frac{0}{a-b}\right|$

where the products of the components of the vector are $c \cdot 0 = 0$ and $c \cdot 0 = 0$, respectively ("dot" product)

The "cross product" of the two vectors is given by the relation:

$\left|\frac{c}{0}\right| \otimes \left|\frac{0}{c}\right| (!\equiv) \left|\frac{c}{0}\right| \odot \left|\frac{0}{c}\right| = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & cc \end{vmatrix}$ ($(!\equiv)$ means "not equivalent"), where

$cc = (a+b) \odot (a-b) = a^2 + b^2 - 2ab$ represents operation of multiplication, where $\odot$ is imaginary, since it is composed of three operations (addition, subtraction, and multiplication), and so the result cc cannot be a group. The same is true for $cc = c^2$ and $cc = c^2$

Note that $a^2 + b^2 \geq 2ab$ but that $c^2 + c^2 = 2c^2$ and $c^2 + c^2 = 2c^2$.

Note that the original matrix of two vectors has no components and a third component (change in dimension) has been created as a third vector with three components $(0, 0, cc)$

Note that $\left|\frac{c}{0}\right| \otimes \left|\frac{0}{c}\right| = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^2 \end{vmatrix}$ and $\left|\frac{c}{0}\right| \otimes \left|\frac{0}{c}\right| = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^2 \end{vmatrix}$ This means that c and c in the definition of the cross product (there are no elements to begin with, so the product does not exist in the first two dimensions.

(This result is an expression of Fermat's last theorem for the case $n = 2$.)

Note that $Det|M_{cc}|$ is not the same as the cross product, since the determinant is relegated to a single dimension, while the dot product between the vector components of $|M_{cc}| \cdot |M_{cc}|$ characterizes the origin $(0,0)$ in two dimensions.

Note that $\begin{vmatrix} c & 0 \\ 0 & c \end{vmatrix}^2 = \begin{vmatrix} (c)^2 & 0 \\ 0 & (c)^2 \end{vmatrix}$ and

$$|M_{cc}|^n = \begin{vmatrix} c & 0 \\ 0 & c \end{vmatrix}^n = \begin{vmatrix} (c)^n & 0 \\ 0 & (c)^n \end{vmatrix} = \begin{vmatrix} (a+b)^n & 0 \\ 0 & (a-b)^n \end{vmatrix}, Tr\left(|M_{cc}|^n\right) = \left((a+b)^n + (a-b)^n\right), Det\left(|M_{cc}|^n\right) = (a+b)^n (a-$$

If $b = 0$, then $c = a \pm 0 = a \;\; c = a$, and for $c = 1$, the matrix representation of two independent unity elements is given by the identity matrix $|I| \triangleq \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ where $|I|^n = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^n = \begin{vmatrix} 1^n & 0 \\ 0 & 1^n \end{vmatrix}$

Note that $2|I| = |I| + |I| = \begin{vmatrix} 1_+ + 1_+ & 0 \\ 0 & 1_+ + 1_+ \end{vmatrix}$ and that

$$\begin{vmatrix} 1+1 & 0 \\ 0 & 1+1 \end{vmatrix}^2 = \begin{matrix} \left[(1_+)^2 + 1_+\right]^2 + 2(1_\times)(1_\times) & 0 \\ 0 & \left[(1_+)^2 + 1_+\right]^2 + 2(1_\times)(1_\times) \end{matrix}$$

Where (1_+) and $(1_\times)$ distinguish between elements that are added and those that are multiplied; if no distinction is made, then $2|I| = \begin{vmatrix} 4(1) & 0 \\ 0 & 4(1) \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix}$

In General

If the restrictions on the Barber and the Villager are removed, then the equations of interaction are given by

$$(c\tau') = (c\tau) + (v\tau')$$
$$(c\tau')^2 = (c\tau)^2 + (v\tau')^2 + 2(c\tau)(v\tau') = (c\tau)^2 + (v\tau')^2 + h^2,\; h^2 = 2(c\tau)(v\tau')$$
$$\pi(c\tau')^2 = \pi(c\tau)^2 + \pi(v\tau')^2 + (c\tau)\left[2\pi(v\tau')\right] = A_r + A_{r'} + r(2\pi r'), h_r^{\,2} = r(2\pi r') = r(C')$$

In the third equation (“radial coordinates”), r is the radius of a circle, and $C' = 2\pi r'$ is the circumference of the second.

Complex representation.

Initially, an existing barber (or set of barbers) is represented by $\sqrt{\{b\}=}\sqrt{\{1\}}$ whether (they) shave (themselves) or not; if (they shave themselves, they are then "positive definite" (real): $\sqrt{b^{2}=}\sqrt{1^{2}}=|1|=\{b\}$ Similarly, for villagers $\sqrt{v^{2}=}\sqrt{1^{2}}=|1|=\{v\}$

If a barber shaves himself, then he is represented by b^2 and similarly for a villager v^2 If there is only one shave per barber or village at a given time t, the sum of shaves can then be given by $n_s = b^2 + v^2$

An existing barber or villager that both shaves and doesn't shave himself can be defined as imaginary (non-existent) by $\sqrt{\{b\}=}\sqrt{\{-1\}}$ or $\sqrt{\{v\}=}\sqrt{\{-1\}}$ so the sum of the results of shaving and not shaving themselves simultaneously is then given by $n_s = b^2 + v^2$

However, if the barbers are defined as real and the villagers as imaginary, so that the set of real and imaginary barbers and villagers can be represented by the complex number $\psi = 1 + i$, where $i = \sqrt{-1}$. Then the events of barbers shaving villagers that do not exist can be represented by complex conjugation $\psi\psi^* = (1+i)(1-i) = 1^2 - i^2 = 1^2 + 1$; note the difference in powers. This means that the operation of existing barbers shaving imaginary villagers ($\pm 1i$) has been eliminated

Note that $i^2 = -1 \neq -1$ since $\log_i(i^2) = 2$ but $\log_{-1}(-1) = \log_1(1) = 1$.

Note that $\psi^2 = (1+i)^2 = \left[1^2 + i^2\right] + 2(1)(i)$ However, the real axis is orthogonal to the imaginary axis, so the vectors in each plane are orthogonal, and the interaction product $2(1)(i) = 2(i)$ does not exist as a real or imaginary number (the dot product is 0). One can imagine the existence of term $\left[1^2 + i^2\right]$ by complex conjugation, where $\psi\psi^* \triangleq (1+i)(1-i) = 1^2 - i^2 = 1^2 + 1$, but note that (again) the exponents of the terms are not equal.

Special Relativity

Note that for

$$\psi = (c\tau) + i(v\tau')$$
$$\psi\psi^* = (c\tau)^2 + (v\tau')^2$$

where $(c\tau')^2 = (c\tau)^2 + (v\tau')^2$ if and only if $(c\tau')^2 = \psi\psi^*$ which is only true if $(v\tau')^2 = 0$ so that $(c\tau')^2 = (c\tau)^2$ (Note that the equation has the form of that of a circle). Nevertheless, relativists soldier on and solve the equation for τ', yielding the so-called "time dilation" equation

$$\tau' = \tau G, G = \frac{1}{\sqrt{1-b^2}}, b = \frac{v}{c}.$$

And if you believe that, I have some shoreline property in a Florida swamp on which Donald Trump wants to build a casino, which I will sell at a bargain price…. ☺ Needless to say, there is much more to this story regarding mathematical physics and the current model, but I don't have the space to write it here. If interested, see the links at the beginning of this document.

(In progress as of 8/25/2021 Stay tuned)

Initial and final states

At the initial and final states there is no interaction $\left(v\tau'\right)^2_i = 0$ $\left(v\tau'\right)_f = 0$ and during the transition from the initial to the final state, $\left(v\tau'\right)^2 > 0$ and the initial state is changing to the final state:

$$\left(c\tau\right)^2_{v\tau'=0} \to \left(c\tau\right)^2 + \left(v\tau'\right)^2 + 2\left(c\tau\right)\left(v\tau'\right) \to \left(c\tau'\right)^2_{v\tau'=0}$$

Let

$$\gamma \triangleq \frac{\tau'}{\tau},\ \beta \triangleq \frac{v}{c}, \gamma \geq \beta, \gamma - \beta = 0 \Leftrightarrow \gamma = \beta$$

Addition is defined as

$$\left(\varphi_{\gamma+\beta}\right)^1 \triangleq \gamma + \beta,\ \log \varphi_{\gamma+\beta}\left(\varphi_{\gamma+\beta}\right) = 1$$

Note that the product $\gamma\beta = \sqrt{\left(\gamma\beta\right)^2}$ (i.e., multiplication) is not defined;.

Multiplication is defined as:

$$\left(\varphi_{\gamma\beta}\right)^2 \triangleq \left(\gamma + \beta\right)^2 = \gamma^2 + \beta^2 + 2\gamma\beta$$

However, the interaction term $h^2 \triangleq 2\gamma\beta$ can be defined so that

$h = \left(\sqrt{h^2}\right) = \sqrt{\left(2\gamma\beta\right)^2} \triangleq \sqrt{\left(2S\right)^2},\ S \triangleq \gamma\beta$ so that $h = \sqrt{2}\left(S\right) \Leftrightarrow S = \dfrac{h}{\sqrt{2}}$

$$\left(\varphi_{\gamma\beta}\right)^2=\gamma^2+\beta^2+h^2=\gamma^2+\beta^2+2S^2$$

Initial State

(Integers)

If $\gamma=n$ and $\beta=m$ are integers then ($n=1,m=0)$ characterizes the initial state where

$$\left(\varphi_\gamma\right)=\sqrt{\gamma}=\sqrt{n}$$
$$\left(\varphi_\gamma\right)^2=\gamma^2=n^2=1$$

The initial state can be characterized by the expression

$\varphi_{c\tau}=1_{c\tau}\triangleq\frac{c\tau}{c\tau},\ \log_{1_{c\tau}}\left(1_{c\tau}\right)=1$, so that $\left(1_{c\tau}\right)^2\triangleq\left(\frac{c\tau}{c\tau}\right)^2,\ \log_{1_{c\tau}}\left(1_{c\tau}\right)^2=2$

A second initial state can be characterized by $n=1,m=m)$ where

$$\left(\varphi_\beta\right)=\sqrt{\beta}=\sqrt{m}$$
$$\left(\varphi_\beta\right)^2=\beta^2=m$$

If the states do not interact (multiply) they can be represented by $|1+m|\triangleq\begin{vmatrix}1 & 0\\ 0 & m\end{vmatrix}$ where

$$|1+m|^p\triangleq\begin{vmatrix}1 & 0\\ 0 & m\end{vmatrix}^p=\begin{vmatrix}1^p & 0\\ 0 & m^p\end{vmatrix}$$

The sum of the initial states is then represented by $\Sigma\triangleq Tr|1+m|=\gamma+\beta=1+m$ where Σ is also an integer.

If multiplication is allowed, then the product Σ^2 is also an integer, where

$$\Sigma^2=\left(\gamma+\beta\right)^2=\gamma^2+\beta^2+\left(2\gamma\beta\right)$$
$$=\left(1+m\right)^2=1^2+m^2+2\left(1\right)\left(m\right)$$
$$=Tr|1+m|+Det\begin{vmatrix}1 & 1\\ -m & m\end{vmatrix}$$

Where $h^2\triangleq2\left(1\right)\left(m\right)=Det\begin{vmatrix}1 & 1\\ -m & m\end{vmatrix}$ is the interaction (multiplication) matrix for integers

Note that for $m=1$ (the first allowable two-state), $\Sigma^2=\left[1^2+1^2\right]+2\left(1^2\right)=$"$4\left(1^2\right)$" where the "existence" term $\left[1^2+1^2\right]$ is distinguished by the multiplicative interaction term $2\left(1^2\right)$ by color.

If the initial state is invariant, then the change in Σ will be given by $\Delta\Sigma = m^2 + 2(1)(m) = m(1+m)$;

$(m = 0) \Leftrightarrow \Sigma^2 = 1^2$).

This can be extended to

$$\mathrm{N} = n + m$$
$$N^2 = n^2 + m^2 + 2nm$$

, where $m = 0 \Leftrightarrow N^2 = n^2$

Hyperbolic integer functions

Note the identification of the term $\Sigma^2 = 1^2 + m^2$ with hyperbolic functions with the identification $\Sigma_\theta{}^2 \equiv \cosh^2\theta$ and $m^2 \equiv \sinh^2\theta$ where the expression can only be obtained as

$\Sigma_\theta \triangleq \cosh\theta + i\sinh\theta$ so that $\Sigma_\theta \Sigma_\theta{}^* \triangleq \cosh^2\theta + \sinh^2\theta = 1^2 + \sinh^2\theta$

Real Numbers (in progess)

(transition period)

Dividing the expression $(c\tau')^2 = \left[(c\tau)^2 + \left(v\tau'\right)^2\right] + 2(c\tau)\left(v\tau'\right)$ by $(c\tau)^2$ characterizes the transition expression as $\left(\gamma\right)^2 = \left(1_{c\tau}\right)^2 + \left(\gamma\beta\right)^2 + 2\left(1_{c\tau}\right)\gamma\beta$, where $\gamma \triangleq \frac{\tau'}{\tau}, \beta \triangleq \frac{v}{c}$, $\gamma \geq 1_{c\tau}, \beta \geq 1_{c\tau}$ and

$$\left(\gamma\right)^2 = \left(\frac{c\tau'}{c\tau}\right)^2 = \left(1_c\right)^2\left(\frac{\tau'}{\tau}\right)^2$$

Note that (γ) is scaled by β so that at the final state, $\left(c\tau\right)$ increases to $\left(c\tau'\right)$ (i.e., $v \uparrow \rightarrow c$ and $\tau \uparrow \rightarrow \tau'$ so that at the final state (just before) $v = 0$

At the final state, $v = c,\ \tau = \tau'$ so that $\left(\gamma\right)^2 = \left(\frac{c\tau'}{c\tau'}\right)^2 = \left[\left(1_c\right)\left(1_{\tau'}\right)\right]^2 = \left(1_{c\tau'}\right)^2 = \left[\left(1_c\right)^2\left(1_{\tau'}\right)^2\right]$,

Where $\log_{1_c}\left(1_c\right)^2 = \log_{1_{\tau'}}\left(1_{\tau'}\right)^2 = 2$ and $\log_{\left[\left(1_c\right)^2\left(1_{\tau'}\right)^2\right]}\left[\left(1_c\right)^2\left(1_{\tau'}\right)^2\right]$

$$\left(1_{\gamma\beta}\right) = \left(\frac{\gamma\beta}{\gamma\beta}\right) \Leftrightarrow \left(1_{\gamma\beta}\right)^2 = \left(\frac{\gamma\beta}{\gamma\beta}\right)^2, \ \left(1_{\tau'}\right)^2 = \left(\frac{\tau'}{\tau'}\right)^2,$$

$$\varphi_{\gamma\beta} \triangleq \gamma + \beta$$

$$\varphi^2 = \left(\gamma\right)^2 = \left(1_1\right)^2 = \left(1_\gamma\right)^2 + \left(1_{\gamma\beta}\right)^2 + 2\left(1_\gamma\right)\left(1_\beta\right)$$

Final State

$$\varphi^2 = \left(1_{c\tau'}\right)^2 = \left[(\gamma)^2 + \left(\gamma\beta\right)^2\right] + 2\left(1_{c\tau}\right)(\gamma)\left(\beta\right)$$

Dividing the expression $(c\tau')^2 = \left[(c\tau)^2 + \left(v\tau'\right)^2\right] + 2(c\tau)\left(v\tau'\right)$ by $(c\tau')^2$ characterizes the transition expression as $\left(1_{c\tau'}\right)^2 = \left[(\frac{1}{\gamma})^2 + \left(\beta\right)^2\right] + 2\left(1_{c\tau'}\right)(\frac{\beta}{\gamma})$, where $\gamma \triangleq \frac{\tau'}{\tau}, \beta \triangleq \frac{v}{c}$, $\gamma \leq 1_{c\tau'}, \beta \leq 1_{c\tau'}$

(Note the identification with $(\frac{1}{\gamma}) \equiv \cos\theta$ and $\beta = \sin\theta$ in the RUC.)